

Inventory buildup and Little's Law

Class handout

[As class starts, Radu passes out a handout with the following case and asks the students to read it out loud as a group, each taking turns to read. Radu gives students some time to ponder and discuss among themselves in smaller groups.]

Milo's dog washing service

Milo ran a mobile dog-washing service called *Paws & Reflect*. On weekends, he parked at the local farmer's market to wash dogs. Milo worked alone. He took one dog at a time into the van, while others waited in line outside.

Milo had been generally well-liked at the farmer's market, but since his business started to pick up, his reception at the market changed. A few nearby vendors began to complain. "Your line is always blocking my booth," one said. Another grumbled that "there are three or four dogs out there every time I look!" Milo wasn't sure if they were exaggerating. He wanted to be a good neighbor, but he also wanted to understand the real situation before making changes.

Since customers sign in when they arrive—first-come, first-served—Milo had accurate arrival times. He also recorded how long each wash took. Here is his data for the first five dogs for a typical Saturday. Milo opens his business at 7:00 am.

Dog	Arrival Time (am)	Wash Time (minutes)
1: Max the Retriever	7:00	4
2: Tofu the Shih Tzu	7:02	3
3: Ginger the Terrier	7:03	2
4: Bear the Bulldog	7:05	5
5: Luna the Poodle	7:07	3

Max was already waiting when Milo opened the van doors. Tofu arrived with his owner, who was late for an outdoor yoga class and was visibly stressed. Ginger showed up barking and tugging on her leash. Bear was a bit grumpy and needed to be coaxed into the van when it was his turn for a wash. Luna trotted up elegantly and seemed happy to wait.

At the end of the fifth wash there was an uncharacteristic lull in traffic so Milo looked at his notes and wondered: "Are these complaints right? Are there really three or four dogs waiting around most of the time?"

Question for consideration:

Based on this collected data, how many dogs are waiting on average in line?

Class discussion

RADU: Let's dive right into the case. Farah, I think you'll like it, there are some calculations to be done!

[All of the students looked eager to discuss, except Ming, who looked more exasperated than usual.]

RADU: Ming, clearly something is on your mind. Tell us.

MING: Well, I guess I find the case so contrived. Who really cares about a dog wash? I didn't come to do an MBA to talk about dog washers.

RADU: You don't have any troubles washing your dog?

MING: What? me? I don't have a dog, but that's beside the point. Shouldn't we talk about real businesses here?

RADU: Milo's business isn't real?

MING: You know what I mean.

RADU: Actually, I do not. Sure, Milo's dog wash is fictional, but I have been to farmers' markets like this many times. They are really a breath of fresh air, are they not? Just because a business is small doesn't mean it's not real. You would rather us talk about Google?

MING: Yes! Can we do that?

RADU: Maybe another time. There really is something to learn here, can I ask for your patience on this Ming? Can we start with a relatively uncomplicated business to learn some of these core concepts?

MING: As long as we do something real later.

RADU: Ming, I think it's interesting you said "something real". By "real", do you mean large?

MING: Emmm, I guess you can interpret it that way. Google is a more important business than dog wash for sure, and is doing some real work in the world. So they are more worthy of our attention.

RADU: It's interesting that you mention large equals importance. I am not sure I agree with you on this. To me, a small business like Milo's serving people in a small neighborhood is also quite meaningful and noble. I promise you will see many different types of reality as time goes. *[Turns to the rest of the class.]* Can we start with the puzzle of this case? How can we learn how many dogs there are waiting in line on average without actually having a count of the dogs? All we have are these times. Does anyone else find this puzzling?

FARAH: No, not really, we have all of the data we need for that.

RADU: You are confident, very good. Can you tell us how it works?

FARAH: It's easy ...

RADU: *[Interrupts Farah.]* Can I interrupt you there? I am happy you can give us your thoughts, but I am not sure you are really in a position to declare that it's easy. I am presuming you found it easy, is that right?

FARAH: Yes, of course.

RADU: Does that mean it is easy for your colleagues here?

FARAH: Well, I just assumed.

RADU: Yes, I understand. Sometimes, when things come easy to us, it is hard to imagine how it could be hard for others. Show of hands, who else here knows how to compute how many dogs are there waiting on average?

[Camila gives a “maybe” shrug, Ming shows indifference, while the two others squint somewhat puzzled.]

RADU: These are your peers. They wrote difficult exams and fought to get in the same school as you. They are not sure how to proceed. Do you still consider this easy?

FARAH: *[Sheepishly.]* I guess maybe I was misjudged. Sorry all.

RADU: *[Smiling at Farah.]* No need to be sorry, Farah. We are all learning here. It is great that you can guide us through this. I am happy you find this easy! The first time I saw something like this, I was lost myself. Where should we start? Can you tell us how you thought about the data in the table?

FARAH: The first thing is to visualize the data over time. We can see how many dogs there are in line if we track when they come and go.

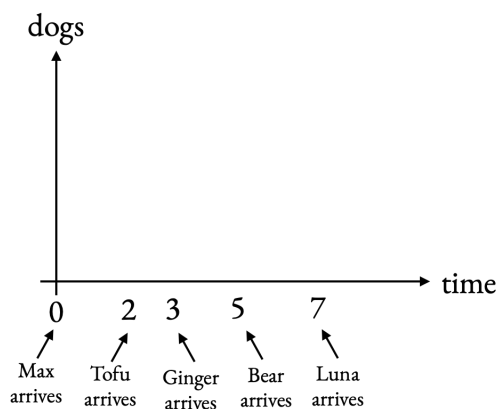
RADU: Ah, yes. It’s like you want to make a figure, with two axes: “time” on the horizontal and “dogs” on the vertical. *[Talks as he draws the following.]*



I made 7:00 am into time “0”, so we can think of the time axis as measuring minutes after 7:00 am. Is this something like what you had in mind?

FARAH: Yes, exactly. And along that time axis, we can place when the dogs arrived.

RADU: Let me do that. *[Turns and writes on the board using data from the table.]*



RADU: [*To Binesi.*] This looks right to you, Binesi?

BINESI: Yes, I think so. You just took the arrival times from the first column of the table in the case and wrote them along the time axis. But so far I am not seeing how we get to this vertical axis. [*Gestures up and down with her hand to the “dog” axis in the figure.*]

RADU: Right, that is my question too. [*To Farah.*] How can we start involving this vertical axis?

FARAH: Sure, let me explain. On the vertical axis, we want to capture how many dogs are in the system. When a dog arrives, we increase this count by 1. When a dog leaves we decrease this count by 1. This is just like in the email you sent after an earlier class.

RADU: Ah yes, very astute. We have their arrival times, I wrote those in. This will give a sense of when the inventory increases. But where do the leaving times come from? [*Turns to Camila.*] Can you help us with that Camila?

CAMILA: Yes, that’s eas...oh wait, I know you don’t like when we say that. [*Sits up in her chair.*] The way you get that is you add the wash time to the arrival time. That is how you get the leaving time.

RADU: Ah, I see your idea. Something like this? [*Draws the following table on the board.*]

Dog	Arr. Time (min)	Wash Time (min)	Leave Time (min)
1: Max	0	4	4
2: Tofu	2	3	5
3: Ginger	3	2	5
4: Bear	5	5	10
5: Luna	7	3	10

FARAH: [*Interrupts while Radu is writing.*] No, no, no, that’s not right.

RADU: Yeah ...I maybe see what you are getting at Farah. I was with Camila for at first, but as I started to write things down I started to question it a bit. Somehow, Tofu and Ginger are both leaving at time 5. That’s a little odd to me. What did we miss here Farah?

FARAH: You need to be careful. Milo can only wash one dog at a time. So while he is washing one dog, the others need to wait. Your last column there assumes there is no waiting. As soon as dogs are arrived they are washed. Only then could you just add the times in those two columns.

RADU: [*To Camila.*] Do you follow Farah’s point here Camila?

CAMILA: Yes, I do. I also realized the issue as soon as you started writing things down.

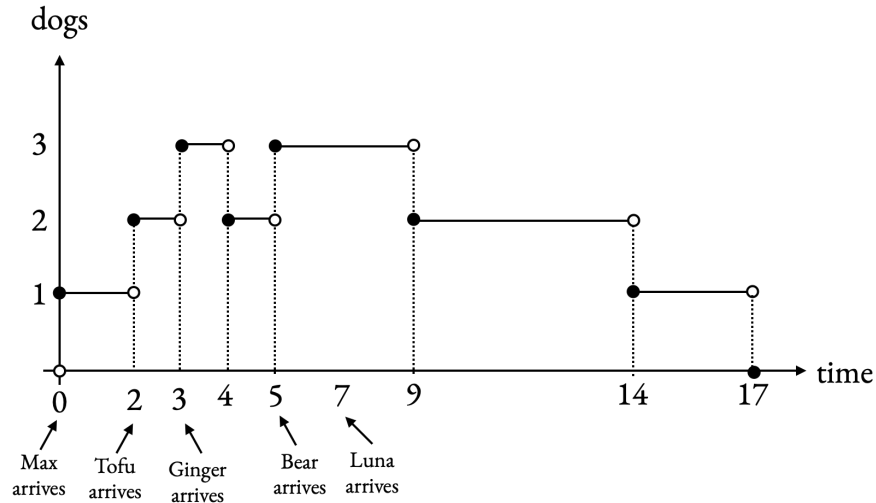
RADU: Then why didn’t you stop me?

CAMILA: Well, truthfully, it was because I just finished saying it was easy. I was a little embarrassed.

RADU: No need to be embarrassed! We are all learning. Can you help us fix this, Camila? I feel Farah knows, but maybe you can bring us all up to speed?

CAMILA: Actually, can I draw something on the figure, I think it will make it easier to explain.

RADU: Certainly, here is the chalk. [*Hands Camila the chalk. She takes a minute to scribble something on a piece of paper and then adds the following lines to figure on the board without saying anything, reading silently from her scribbled piece of paper.*]



RADU: Wow, I am very impressed with this Camila. But may I ask you, why did you silently solve it on paper and then just write it up here without explaining how you were getting this?

CAMILA: Well, I just wanted to write the correct diagram here, what is there to explain?

RADU: Indeed, indeed. Correctness explains itself possibly? But, what if others cannot follow you? What is the value of being correct if others do not understand what you have done?

CAMILA: Isn't it up to them to convince themselves it's correct?

RADU: Maybe, but since we are in a classroom, why don't we ask them? *[Turns to the other students.]*

Does everyone understand what Camila has drawn for us here?

[There a short silence and everyone stares at the board. Then Haoyu speaks.]

HAOYU: I can say that I am completely confused.

RADU: Good! So what question do you have for Camila?

HAOYU: I hadn't formulated a question, I was just confused.

RADU: Then, please take a moment. Confusion is a natural state, but we can only get out of it if we have a question to ask. Take a minute to think about what you want to ask.

[Haoyu sits up in his chair and starts to ponder. Meanwhile Camila starts to slink away from the blackboard back towards her desk.]

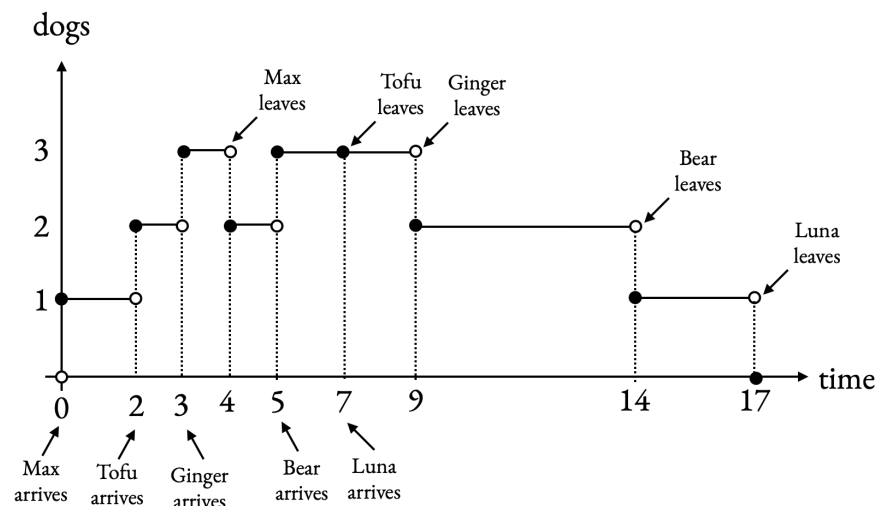
Camila, you stay up there. You are answering Haoyu's questions, are you not?

CAMILA: *[Turns to Radu looking startled]* Oh, me? Well I can try. You don't think it's right?

RADU: I didn't say it was right or wrong, but I think you are in the best position to answer questions about what you have done, are you not? I mean, how can I answer a question about what *you* were thinking? *[Turns to Haoyu.]* Haoyu, you are ready with your question?

HAOYU: *[Clears his throat.]* Yes, I guess I am not sure about the jumps and drops in the graph. I understand when a dog arrives to the line, the graph jumps. And I understand that when a dog leaves, the graph drops down. But I only see four jumps and four drops, but there are five dogs. So what happened? To be honest, I still haven't made sense of the numbers, I guess my confusion is more fundamental than any numbers.

CAMILA: Oh, I think I can explain that potential confusion. Let me label the graph with the points that the dogs leave. *[Turns and adds more to the picture]*



HAOYU: So, wait. Tofu leaves at time 7 and there is no drop?

CAMILA: Yes, because Luna arrives at the same time.

HAOYU: Oh ... that's tricky. One dog leaves at exactly the same time another arrives. I didn't think of that.

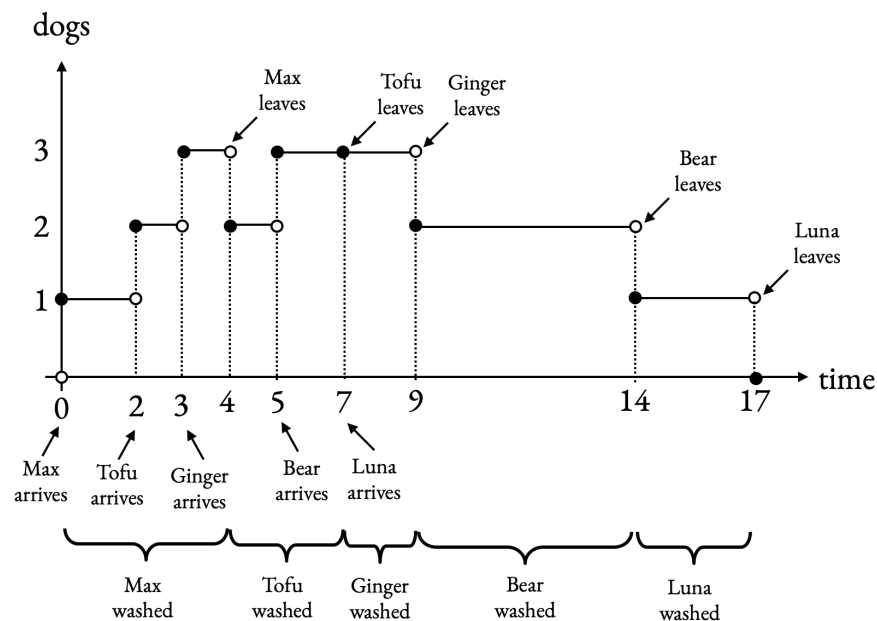
BINESI: *[Looking at Radu]* May I ask ask a question?

RADU: *[Pointing towards Camila]* Ask her, not me.

CAMILA: *[Now looking directly at Binesi.]* Sure, go ahead.

BINESI: I guess I am just having a hard time understanding what is happening in the figure. For example, where can I see when the dogs are being washed. Milo works the whole time washing dogs, how do I see that in the figure?

CAMILA: Oh sorry, I thought that might have been clear. Let me draw more here. *[Turns to the figure and adds something to the bottom]*



Does that clear thing up?

BINESI: Oh, hmmm. *[Still looks confused but does her best to show she understands]*

RADU: Camila, it might help if you explain one of these in words for us. You seem to have a very strong visual grasp of what is happening, but maybe can you connect things in words for us? How do you get these brackets? It might help us be with you more. How did you get that Ginger, say, is washed from time 7 to 9? After all, she arrives at time 3?

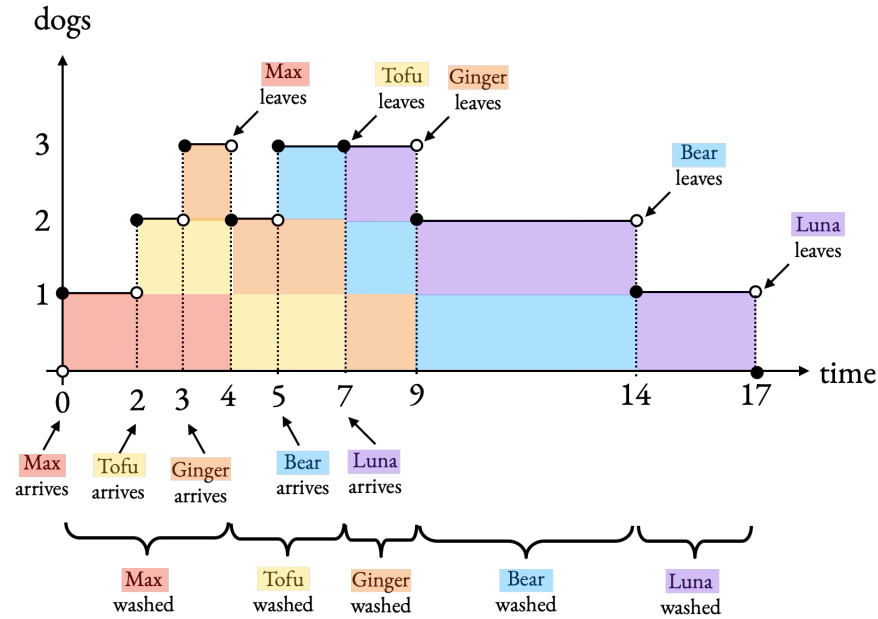
CAMILA: Well, as the picture shows, Max is being washed from time he arrives until he leaves. It takes 4 minutes to wash Max and so he leaves at time 4. So even though Ginger arrives at time 3, Milo cannot work on her right away. Ginger needs to wait for Max to finish, and even for Tofu to finish before her because it's first-come first-served. Milo starts washing Tofu right when Max leaves at time 4. Since it takes three minutes to wash Tofu, Tofu is done at time 7. Only when Tofu leaves can Milo start washing Ginger. This takes from time 7 to time 9. Does that clear things up? *[Looking at Radu, who points Camila to Binesi]* Sorry, Binesi, does that clear things up for you?

BINESI: That certainly helps, but ... *[Still looking confused.]*

FARAH: Camila, do you mind if I add something to the figure that I think might help everyone?

CAMILA: Yes, of course. *[Hands Farah the chalk and eagerly sits down]*

FARAH: I think what can help is to indicate the "journey" of the dogs in the figure. I can use the colored chalk to do that. *[Takes five pieces of colored chalk and colors the figure]*



If the dog's color is near the bottom of the figure (between 0 and 1), it means Milo is washing them during that time. The higher is their color in the figure, the more back they are in the line. I thought that might help.

RADU: Oh, I like this very much. Take Luna (purple) for example. She arrives in the system at time 7 as soon as Tofu (yellow) leaves. But Ginger and Bear are both ahead of her. So she has to wait for both of them to finish before she can be washed. Washing Ginger takes 2 minutes and washing Bear takes 5 more minutes. So not until 7 minutes after Luna arrives (at time 14) does Milo start washing Luna. Then it takes three more minutes for Milo to wash Luna, and so she leaves at time 17. We can see all of this in the purple color!

FARAH: [*Excitedly*] Yes, that is exactly how it works. Isn't it amazing?

RADU: It is a marvel indeed, provided you understand it. Binesi, how does it feel now?

BINESI: Clearer than ever, but I am still processing. I am not sure if this is the right way to look at the picture. I look at the different color boxes and I think of each of them as a dog crate. When a dog arrives, its crate is put at the highest level. The dog at level 1 get washed by Milo. So each time a dog leaves, all waiting dogs crate get one level lower. Now I can get the "flow" and "dynamic" side of the picture.

RADU: Great imagination and great way to look at the picture, Binesi. Nice job!

Let me test you all a bit on your understanding; it may help everyone clarify. Let's take our total time to be 17 minutes, the time at which all five dogs have left. What is the average throughput during this period?

MING: This is easy, and we don't need the picture for it. Five dogs were served and so the throughput is $5/17$ dogs per minute.

RADU: Right, 5 dogs in 17 minutes. But I don't agree that you don't need the picture to see this. How else would you know that 17 is the time at which all five had come *and gone*?

MING: [*Pauses and thinks for a few seconds.*] Okay, point taken.

RADU: Now for something more difficult. What is the average flow time during this period?

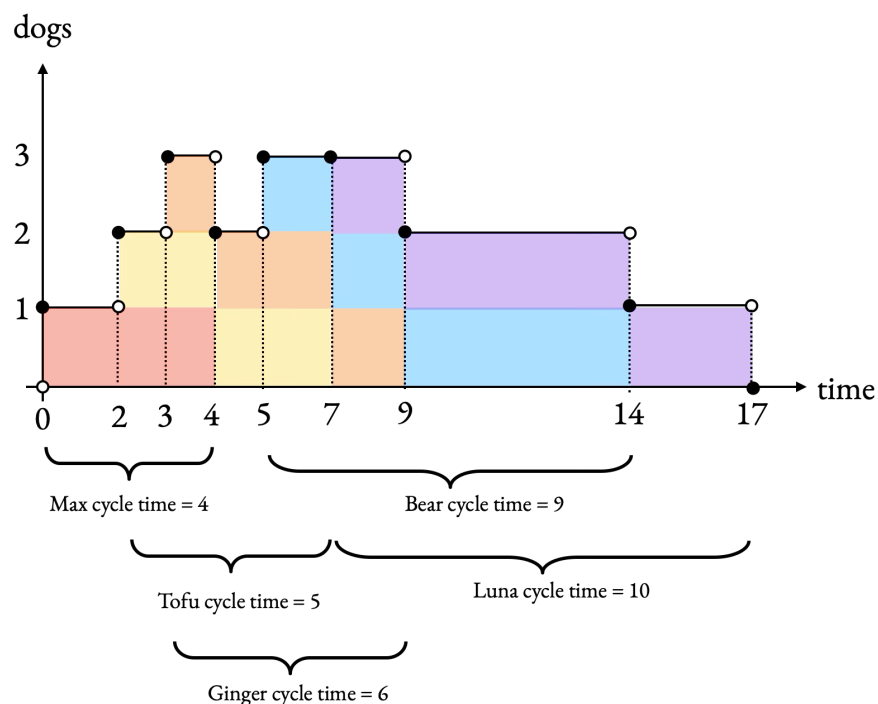
[The students take longer to respond.]

FARAH: $34/5$ [Not looking up from her paper with calculations.]

RADU: [Laughing] Farah, you know by now I am not going to be happy with a blurted out number with no units and no explanation, am I?

FARAH: Right, sorry, force of habit. By the definition of average flow time, we need to first sum up the flow time of all dogs and take the average. There are five dogs, that is why we divide by 5. The number 34 is the sum of the five individual flow times of the dog. To see the individual flow times you just look at the colors in the figure. Max is in there for 4 minutes, Tofu is there from time 2 to 7, for a total of 5 minutes. Ginger takes 6 minutes, and Bear enters at time 5 and leaves at time 15, so 9 minutes. Luna shows up the latest and has to wait a long time indeed. She arrives at time 7 and leaves at time 17, for a total of 10 minutes. Thus, the sum of the flow times is $4 + 5 + 6 + 9 + 10 = 34$ minutes. So the average flow time is $34/5$ minutes.

[As Farah speaks, Radu follows along by adding the following details to the figure]



Very good, and well explained. Is anyone here surprised that the later a dog arrives to the system, the longer they are in the system?

HAOYU: Not at all. I experience this all the time when I go to the barber. The later in the day I arrive, the longer I have to wait.

RADU: Indeed, and why is that? Did what we discussed here today give you new insight into that?

HAOYU: In a way. The reason you wait longer if you arrive later is that you have to wait out the time it takes for everyone ahead of you to get their haircut. If you arrive later, the more

people ahead of you in line. Your wait gets multiplied by everyone ahead of you in line. We see that with Luna in the picture. She needs to wait out Ginger and Bear's washing times before she can be served.

RADU: I like how you said "multiplied". As we will see in classes ahead, the wait time indeed "compounds" like this in busy systems. [*Turns to the whole class.*] That leaves one final performance metric, and arguably the most relevant one to the question of the case: average inventory. Any ideas how we approach this?

FARAH: I think I can do it.

RADU: I am quite sure you can [*Smiles widely*]. Shall we give someone else a try Farah?

FARAH: Oh yes, of course.

RADU: Camila, you got us into whole way of thinking. Does your figure help us with this average inventory thing?

CAMILA: Well, it should be something between 1 and 3. At all points of time in 0 to 17, there are between 1 and 3 dogs, as the figure indicates on that vertical axis there [*Makes an up and down gesture with her hand*]. It looks like the mode inventory is 2, and the times at which there are 1 and 3 kind of "balance" each other out. That is, an equal percentage of the time there is 1 dog as there is three dogs ... five minutes it looks like. So I am thinking the average number of dogs over time is 2.

RADU: You nailed it! That is amazing. The average inventory is, in fact, 2. Let me include in my email after class a more systematic way to see this, but since we are low on time to cover this now, is everyone okay with Camila's answer here? [*Everyone nods.*] So, right, we have an average throughput of 5/17 dogs per minute, an average flow time of 34/5 minutes, and an average inventory of 2 dogs. We calculated all of our metrics from the simple table of data provided in the case. So, can we use this data to conclude if Milo has a problem with too many dogs in his line?

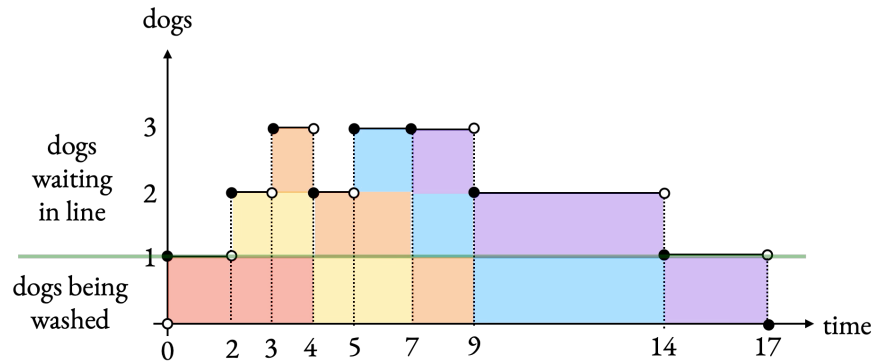
MING: Assuming that this is a typical day, sure. It appears the answer is no. There is on average one dog waiting in his line, and never more than two. His neighbors at the farmer's market are just being grumpy.

BINESI: Wait, am I missing something? Isn't the average number of dogs two?

MING: Yes, two in total. But one is inside with Milo being washed, so we don't count that one as being in the line.

BINESI: Oh, right, right, right.

RADU: Great question and great answer. We can see that in the picture. [*Puts the finishing touches on the figure.*]



This completes our analysis of the case, but I want to leave with one more insight in the time that remains. Look again at the numbers: throughput of $5/17$ dogs per minute, flow time of $34/5$ minutes, and inventory of 2 dogs. Does anyone see any relationships between these numbers: $5/17$, $34/5$, and 2?

FARAH: Yes, it's obvious! $2 = \frac{5}{17} \cdot \frac{34}{5}$. And the units cancel out too. It's very cool!

RADU: I agree that it is cool, but, again, why use the word "obvious"? I would say it is far from obvious, no?

FARAH: Force of habit again ...

RADU: Farah, I think we can all see that you have a lot of insight for these things. Is it possible for you to accept that these skills you have developed are not necessarily obvious to others, and so there is no need to use words like "easy" and "obvious" when making your contributions here?

[Farah gets a bit awkward and her face turns red.]

FARAH: I will certainly try, my apologies. But I want to clarify that I didn't mean to make other students feel bad. Just a habit.

RADU: No need to apologize, Farah. We know you didn't mean it. But let me say that your insight here is one of the most important ones in the whole course! In fact, it is called *Little's Law*.

[Writes on the board]

Little's Law : $I = RT$.

Recall that I is the average inventory, R is the average throughput, and T is the average flow time. If you plug in the numbers we calculated, it checks out, just as Farah just did.

Little's Law : $I = RT$ $2 \text{ dogs} = \frac{5 \text{ dogs}}{17 \text{ minutes}} \frac{34}{5} \text{ minutes}$
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Since Farah discovered it for us here, I may also call it Farah's Law.

[Some students looked at Farah in smile and Farah gets more excited..]

FARAH: Wait, you are saying this always works? It wasn't just an accident these numbers worked out?

RADU: It seems Farah hasn't realized the power and importance of what she just proposed. "Always" is a dangerous word, there are always some exceptions, but let me say this. Little's Law works well for almost every practical scenario you are likely to come across, provided you think about it correctly. It is a simple formula with deceptive power. Let's continue our exploration of it next class.

Email

I wanted to follow up after class with a short written summary of what we covered, and perhaps more importantly, why we covered it the way we did. Several of the ideas from today's discussion are simple in form but subtle in meaning, and it is normal for them to take time to settle.

Calculating average inventory

I promised to show *systematically* how Camila arrived at an average inventory of 2 dogs. Recall the area formula for average inventory that I discussed in the previous email:

$$\text{Average inventory} = \frac{\text{area under the inventory buildup diagram}}{\text{total time}}.$$

This is exactly the tool we need here, and it is worth seeing it applied to a "messier" example than the one we used last time.

Look again at the final version of our figure for Milo's morning, the one with all five dogs colored in. The cleanest way to find the area is to break the 17-minute period into intervals during which the inventory level stays constant, and then add up the areas of the resulting rectangles—each rectangle has a width (the length of the interval) and a height (the inventory level during that interval). Concretely:

- From $t = 0$ to $t = 2$ (2 minutes): only Max has arrived. Inventory = 1.
- From $t = 2$ to $t = 3$ (1 minute): Tofu has arrived; Max and Tofu are both in the system. Inventory = 2.
- From $t = 3$ to $t = 4$ (1 minute): Ginger has arrived; Max, Tofu, and Ginger are all in the system. Inventory = 3.
- From $t = 4$ to $t = 5$ (1 minute): Max has left. Tofu and Ginger remain. Inventory = 2.
- From $t = 5$ to $t = 7$ (2 minutes): Bear has arrived. Tofu, Ginger, and Bear are in the system. Inventory = 3.
- From $t = 7$ to $t = 9$ (2 minutes): Tofu leaves at exactly the moment Luna arrives, so the inventory level does not change. Ginger, Bear, and Luna remain. Inventory = 3.
- From $t = 9$ to $t = 14$ (5 minutes): Ginger has left. Bear and Luna remain—Bear is being washed, Luna is waiting. Inventory = 2.
- From $t = 14$ to $t = 17$ (3 minutes): Bear has left. Only Luna remains, now finally being washed. Inventory = 1.

Adding up the areas of these rectangles (width times height) in each case gives

$$(2 \times 1) + (1 \times 2) + (1 \times 3) + (1 \times 2) + (2 \times 3) + (2 \times 3) + (5 \times 2) + (3 \times 1).$$

Computing term by term:

$$2 + 2 + 3 + 2 + 6 + 6 + 10 + 3 = 34.$$

So the total area under the inventory curve is 34 dog-minutes. Dividing by the total time, $\tau = 17$ minutes, gives

$$\text{Average inventory} = \frac{34 \text{ dog-minutes}}{17 \text{ minutes}} = 2 \text{ dogs.}$$

This confirms what Camila found by eye: the average inventory really is 2 dogs.

Little's Law and what the units are really saying

We ended class by observing a relationship between three quantities:

$$I = RT,$$

where I is average inventory, R is average throughput rate, and T is average flow time. This relationship is known as *Little's Law*.

A word about the word “law.” This is the first thing in our course that I am going to call a law, and the word deserves a small pause, because it is a grander word than it first appears. In our everyday speech, a “law” is a rule someone made up and could just as easily unmake: the speed limit is 50 because a city council voted for 50, and next year they might vote for 60. The “laws” we mean here are not like that. Nobody decided Little's Law. It was not passed, and it cannot be repealed. It is closer to what we mean when we say “the law of gravity”: it is not a rule imposed on the world, but a pattern discovered in it, something that was already true long before anyone wrote it down and would go on being true even if we all forgot it. When I tell you that average inventory equals throughput rate times average flow time, I am not telling you how I would like your process to behave. I am telling you how it already does behave, whether you like it or not, whether you have ever drawn a diagram or not. That is the strange power of a law: it holds even in the operations of people who have never heard of it.

You will also see these truths go by other names, and the names are worth knowing so they don't intimidate you. Sometimes we call them “theorems”, which is the word mathematicians prefer for a statement that has been proved, i.e., established beyond argument by pure reasoning, like the way Little's Law was proved in 1961. Sometimes you will meet a “lemma”, which is a humbler creature: a small truth that is not very exciting on its own, proved mainly because it is a stepping stone to a bigger one. “Theorem, lemma, law”, do not let the vocabulary put you off. Underneath all three is the same simple and rather wonderful idea: some things are always true. If we are careful enough, we can find out what they are and lean on them.

I find myself oscillating between two views of this formula. On some days, it feels deeply profound. On others, it feels almost embarrassingly obvious—perhaps not even deserving the name “law.” Let us begin with the “obvious” interpretation.

In introductory physics, one learns the relationship

$$d = vt,$$

where distance is measured in meters, velocity in meters per second, and time in seconds. The units tell the entire story:

$$\text{meters} = \frac{\text{meters}}{\text{second}} \times \cancel{\text{seconds}}.$$

There are not many other sensible ways to combine these quantities so that the units come out correctly.

Little's Law has the same structure:

$$\text{jobs} = \frac{\text{jobs}}{\text{minute}} \times \cancel{\text{minutes}}.$$

In that sense, the formula is almost forced upon us by consistency of the units.

But this analogy quickly becomes unsettling. Throughput rate feels like a *speed*. Flow time feels like a *time*. Yet inventory does not obviously resemble a distance. In what sense is inventory a “length”? Personally, I do not see it immediately.

Imagine a perfectly stable conveyor belt producing cans of dog food. Suppose exactly two cans enter and leave the belt every second, so the throughput rate is

$$R = 2 \text{ cans/second}.$$

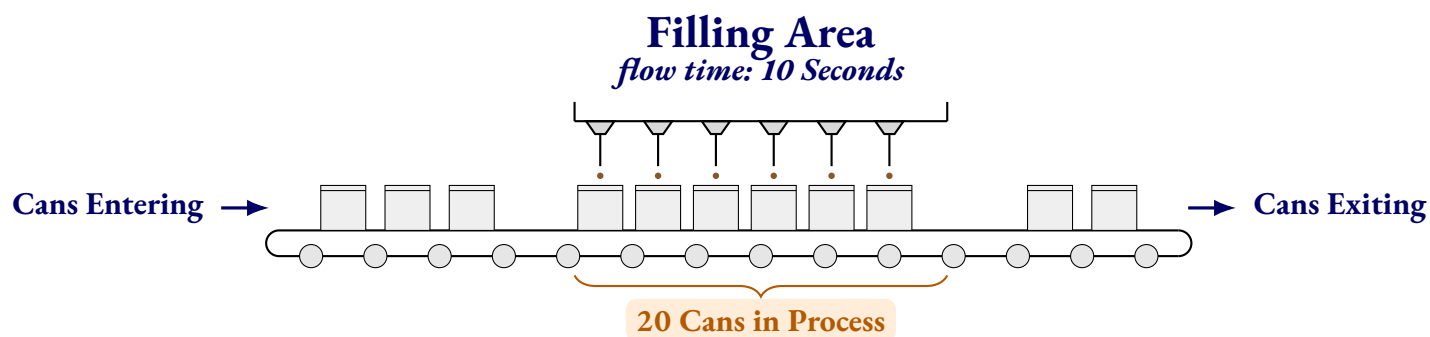
Suppose further that every can takes exactly ten seconds to travel from start to finish, so the flow time is

$$T = 10 \text{ seconds}.$$

How many cans are on the conveyor belt at any given moment? The answer is

$$I = 2 \times 10 = 20 \text{ cans}.$$

In a very real sense, the conveyor belt is “20 cans long.” Here, distance is being measured not in meters, but in cans.



This example is deliberately simple and unusually smooth. Yet the remarkable fact about Little's Law is that it continues to hold *on average* even when systems are irregular, noisy, and human. That is where its true power lies—and also why it deserves our continued attention.

A short history of Little's Law.

It is worth pausing on the history of this relationship, because it illustrates something we have returned to a few times this semester: the same idea can strike different people, at different times, as either entirely obvious or genuinely difficult to pin down.

Before 1961, the relationship $I = RT$ was something like the status of a “folk theorem”. It was a fact that experienced practitioners knew to be true, that kept appearing whenever you worked through a particular example, but that no one had shown had to be true in general. One of the earliest appearances in print is in a 1954 paper by Alan Cobham on priority rules for waiting lines, where the relationship is simply used, as a known fact, without comment or proof [1].

A few years later, the physicist-turned-operations-researcher Philip M. Morse published an influential textbook, *Queues, Inventories and Maintenance* [4]. Morse worked through the mathematics of many different queuing models—different arrival patterns, different numbers of servers, different service rules—and noticed that, model after model, this same relationship between the average number of items in the system, the rate at which items arrived, and the average time an item spent in the system kept reappearing. Morse pointed this out repeatedly as he went, but he did not prove that it had to hold in general. Instead, somewhat in the spirit of someone who has noticed a pattern but is wary of overclaiming, he more or less invited his readers to go looking for the circumstances under which the relationship might *fail*—to find a counterexample, if one could be found, rather than to assume the pattern was a theorem.

This is where John D. C. Little enters the story. By a pleasing coincidence of academic genealogy, Little had been a doctoral student of Morse’s at MIT a few years earlier. In the late 1950s, Little was teaching a course on queuing at the Case Institute of Technology, using Morse’s book as the course text. In his own retrospective account written fifty years later, Little recalls making, in class, more or less the same observation Morse had made in the book: that this relationship “seemed very general” [3]. After class, a student—Little names him Sid Hess—asked Little directly: how hard would it be to prove this in general? Little recalls answering, on the spur of the moment, that it probably wouldn’t be too hard. “Famous last words,” as Little puts it.

The result, published in 1961 in the journal *Operations Research* under the unassuming title “A proof for the queuing formula: $L = \lambda W$,” [2] gave the first general proof of the relationship. The notation L , λ , and W is the queuing-theory notation of the time— L for the average number in the system (our I), λ for the average arrival rate (our R), and W for the average time spent waiting in the system (our T)—but the relationship is exactly the one we have been calling $I = RT$.

I find this small history rather charming, for at least two reasons. First, it is a nice illustration of the gap between *noticing* a pattern and *proving* that the pattern cannot fail. Morse noticed the pattern, repeatedly, across many different models, and even pointed his readers toward the boundary of where it might break down—but stopped short of claiming it as a general law. Little showed why, under reasonably general conditions, it cannot fail. These are different kinds of contributions, and it is interesting, if perhaps a little unfair to Morse and to Cobham before him, that it is the second kind of contribution that tends to get the law named after it.

Second, the history is a nice illustration of something we discussed when Farah “discovered” this relationship for us in class. The very same fact can feel obvious to one person and yet be the object of real intellectual effort for another. By Little’s own account, he treated the relationship as something almost throwaway—a passing remark in a lecture, the kind of thing one says without quite expecting to be asked

to defend it. It was a student, not a professor, who insisted that “seems very general” and “is true in general” are not the same thing, and that the gap between them was worth the effort of closing.

A word of caution: balance matters.

There is, however, an important piece of fine print attached to Little’s Law that is easy to miss, and that I want to flag now even though we will not dwell on it.

Little’s Law, as we have stated it, $I = RT$, implicitly assumes that the system is, in some sense, *balanced*: that, averaged over the period in question, the rate at which jobs arrive equals the rate at which jobs leave. If a system is steadily filling up—more arriving than leaving, period after period—then “the” throughput rate R is not really a single well-defined number, since the arrival rate and the departure rate are different and the inventory I is growing without bound. Little’s Law, in its simplest form, is a statement about systems in a kind of steady state, observed over a long enough or stable enough period that arrivals and departures balance out.

This is not a fatal restriction—many real systems are approximately balanced over the periods we care about, and Milo’s morning, for instance, ends with all five dogs having both arrived and left, so arrivals and departures are perfectly balanced over our 17-minute window. But it is worth keeping in the back of your mind. If you ever find Little’s Law giving you a strange or nonsensical answer, the first thing to check is whether the system you are looking at is actually in this kind of balance.

References

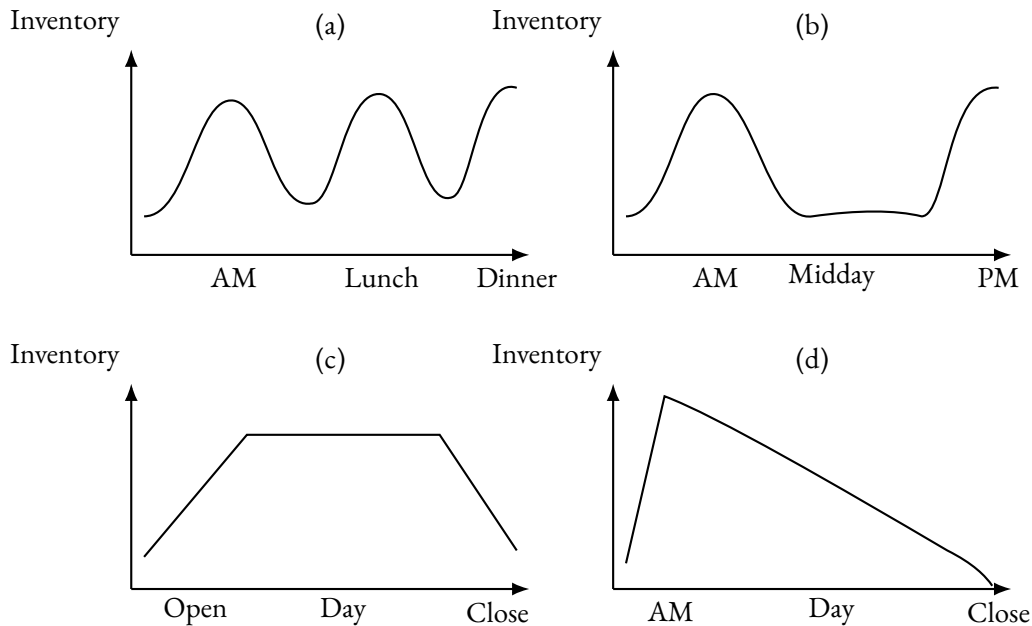
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Extensions

[Posted as homework on the course’s website for this class session.]

1. Milo hires a second dog washer named Hubert to assist him. Hubert has his own work station and can wash a different dog at the same time as Milo. Both can still only wash one dog at a time. How would the inventory buildup diagram of the day look different in this scenario, assuming the same arrival times and wash times described in the case? What if instead Hubert’s wash time is twice that of Milo for each dog?

2. As a student, how can one think of unfinished homework assignments as inventory? How does your homework buildup diagram look like on a typical week? How would the graph differ between someone who procrastinates and someone who doesn't? How would you be able to tell the difference?
3. Describe an operation that might have an inventory buildup diagram throughout the day that has the following structures:



- (a) peaks in morning, lunch, and dinner
 - (b) peaks in morning and evenings, but not during the day
 - (c) a ramp up in the morning, smooth for the rest of the day, and a ramp down at the end of the day
 - (d) a big spike in the morning and then slowly goes down until empty
4. Many businesses track an amount called *accounts receivable*: money that customers owe the business for goods or services already delivered, but not yet paid for. A business might deliver a product on Monday but not receive payment until thirty days later, and during those thirty days, that unpaid invoice sits on the books as a receivable.

Without doing any calculations, think about accounts receivable through the lens of an inventory buildup diagram.

- What is the “job” in this setting? What does it mean for a job to “arrive” into the system, and what does it mean for a job to “leave”?
- What would the average “inventory” in this diagram correspond to, in terms a finance manager might recognize?

- What would the average “flow time” correspond to? Have you heard a term like this used in a business or accounting context before?
 - If a company wanted to reduce its average accounts receivable, what are the different “levers” it could pull, in terms of the components of Little’s Law?
5. Do you know systems whose inventory keeps piling up and this leads to failure? What is the reason for it?

Reflections

[Posted as a reflection exercise on the course’s website for students to write about in a journal due at the end of semester.]

1. What are you doing “on average” during the day? Why is this question difficult to answer? What does “average” mean here?
2. Do you prefer taking window seats or aisle seats on a plane? Describe the difference in these choices from an inventory perspective with respect to the process of someone needing to use the restroom.
3. Think of something in your life that you would describe as “piling up”: unread emails, unfolded laundry, unanswered messages. Try to sketch, roughly, what its inventory buildup diagram would look like over the course of a week. Does it ever reach zero?
4. Before 1961, the relationship we now call Little’s Law was something experienced researchers *believed* to be true—they had seen it work, again and again, in case after case—without anyone having shown that it *had* to be true. Can you think of something in your own work, studies, or life that you have come to believe is “always true” on the strength of repeated experience, but that you have never actually tried to justify or prove? How confident are you in it?