

Inventory-related costs

Class handout

[As class starts, Radu passes out a handout with the following case and asks the students to read it out loud as a group, each taking turns to read. Radu gives students some time to ponder and discuss among themselves in smaller groups.]

Case: Tax Season at Marchetti & Singh

Marchetti & Singh was a small accounting firm tucked above a pharmacy on a quiet commercial street. For most of the year it was a calm place, with Billy Marchetti and a couple of year-round staff handling bookkeeping, payroll, and the steady trickle of clients who needed something filed. But for a few weeks each spring, as the tax deadline approached, the office transformed.

Tax season was, by a wide margin, the firm's biggest time of year. To handle the surge, Billy hired five accounting interns every spring. These interns were usually students from the local college, eager and capable, and brought on for the final push before the deadline. After careful training by Billy, each intern could prepare about 10 returns per day on average. Billy himself did not prepare returns during the crush; instead, he kept the rest of the business running and personally attended to helping the interns with the tax preparation process.

Clients arrived steadily, some in person clutching folders of receipts, others sending their documents over email. Billy tracked when each request came in. The pattern was familiar to him by now. In the twenty-day stretch running from 35 days before the deadline to 15 days before it, new return requests arrived at about 55 per day. Then, in the ten-day stretch from 15 days before the deadline to 5 days before it, the pace quickened to about 80 per day, as procrastinators and the newly anxious flooded in. Five days before the deadline, the firm stopped accepting new requests altogether—a hard cutoff Billy had instituted years ago, posted on the door and in the email auto-reply.

Period	Days before deadline	New requests per day
Early season	35 to 15 (20 days)	55
Late season	15 to 5 (10 days)	80
Cutoff	5 to 0 (5 days)	0

Marchetti & Singh had built its reputation partly on a promise that few competitors would make. If the firm failed to submit a client's return by the deadline, it covered the cost. Specifically, the firm paid the \$50 extension filing fee on the client's behalf, plus an additional rate of \$10 for each day the return was late. It was a point of pride for Billy—a way of telling clients that their deadline was his deadline.

But pride has a price, and Billy had never sat down to figure out what it was. The interns worked hard, the office hummed, and somehow every spring a stack of returns always seemed to go out

after the deadline had passed. Billy wondered how bad it really was.

Question for consideration:

In a typical tax season, how much does Marchetti & Singh pay in extension filing fees and late fees?

Class discussion

RADU: Let's get right to Billy's question. He wants to know what his promise costs him in a typical season. Farah, looks like you've already started.

FARAH: I have. The penalty has two pieces, so I need two numbers. There's 50 dollars for each extension filing, and then 10 dollars for each day a return is late. So I need to know how many returns need extensions, and how many days late they are.

RADU: Good. Start with the first?

FARAH: For that I need to know how fast the office works against how many jobs arrive in total.

RADU: Why these two numbers?

FARAH: To find the number of extensions filed we need to know how many returns are in the office unsubmitted at the deadline.

RADU: Sure, shall we call that value E for now, indicating the number of extensions filed?

FARAH: Sure, let's call it E . So once we have that number, the late extension fee cost to Billy is $50E$ dollars.

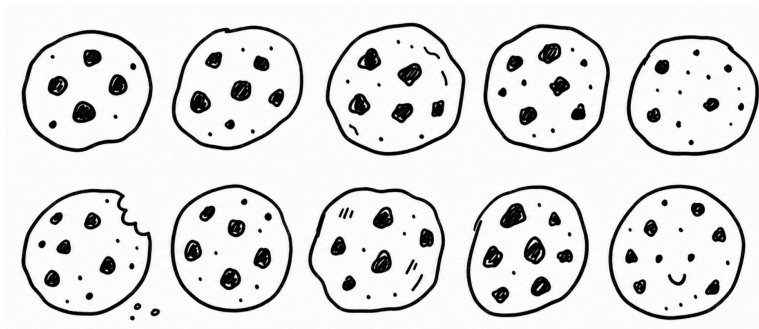
RADU: Right, 50 dollars per extension. So how do you go about getting this E ? You said you need two numbers?

FARAH: Yes, we need to know the total number of returns processed throughout the whole season. If we divide this by the rate at which they process returns, we get E .

RADU: Divide to get E ?

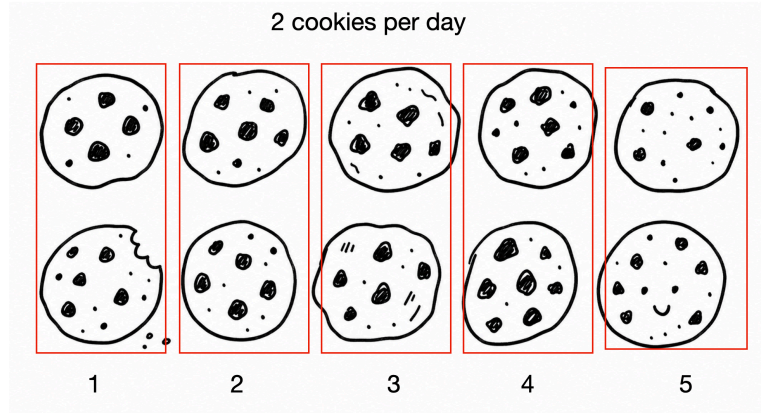
CAMILA: Yes, I see it. Think about it this way. Suppose you have a box of 10 cookies.

[Radu nods and draws 10 cookies on the board.]



CAMILA: Right, and suppose you eat 2 cookies a day.

[Radu draws circles around groups of 2 cookies.]



RADU: Yes, of course, then you will finish over 5 days. We have, 10 divided by 2 equals 5. So if we take the total number of returns and divide by the daily rate of finishing returns, this will give us ...the number of days to finish all of the returns.

CAMILA: Precisely.

RADU: But this is not E , right? We used E to denote the extension filings. Farah, where did you take us?

FARAH: Can't I just compute the answer and show you it works?

RADU: Hmm, I don't like the sound of that, but I have faith, so let's try.

FARAH: Let's start with the rate the office processes returns. This is like the rate Camila eats cookies.

CAMILA: [*Interrupting*] I don't eat *that* many cookies! It was just an example.

FARAH: [*Not responding to Camila*] There are 5 interns, each can do 10 returns per day, so the office finishes 50 returns a day.

[*Radu writes as Farah speaks.*]

$$C_{\text{office}} = 5 \text{ interns} \cdot \frac{10 \text{ returns}}{\text{day}} = 50 \frac{\text{returns}}{\text{day}}$$

RADU: Wait just a minute here, my units are not working out. I can't cancel anything! The units are interns times returns per day on the left here. [*Points at the left side of the equation he just wrote on the board.*]

FARAH: Hmm, I am less concerned about units than you, I just know these numbers work out. Can we keep going? This is taking so long to compute something so trivial. [*Looking exasperated.*]

RADU: Now just hold on Farah. I am already going with you on this computation journey and you have not convinced me of the value of where we are headed. So you need to be a little patient.

FARAH: You are right, I'm sorry.

RADU: No need to be sorry, just let me think here. I am generally less concerned about the numbers than the units. I feel if my units are not working, I am missing something. What am I missing here?

CAMILA: You *can* cancel the “interns” unit.

RADU: Really, how?

CAMILA: Well, the unit on the number 10 there is actually 10 returns per day per intern.

RADU: Oh ... like this?

$$\frac{10 \text{ returns}}{\text{day} \cdot \text{intern}}$$

CAMILA: Exactly, now things cancel.

[Radu adjusts the equation on the board to verify.]

$$C_{\text{office}} = 5 \cancel{\text{interns}} \cdot \frac{10 \text{ returns}}{\text{day} \cdot \cancel{\text{intern}}} = 50 \frac{\text{returns}}{\text{day}}$$

RADU: Thanks Camila, this is very clarifying. And actually foreshadows something we will look at later. *[Pauses to regroup himself.]*

FARAH: *[Quietly outside of Radu’s hearing]* If we ever get through this initial calculation.

RADU: So, Farah, sorry, let’s get back to you. So we have the rate of processing returns, or the rate of Camila’s cookie eating.

[Camila’s throws up her hands again, but can’t help but start to laugh when Haoyu cracks open a box of cookies and starts eating.]

Now we need to know the total number of cookies. How do we compute that?

CAMILA: Can we get back to talking about returns and not cookies?

[Haoyu takes a big bite of a cookie just after Camila finishes her question.]

RADU: Yes, Farah. Bring us back to counting the number of returns.

FARAH: The returns come in across two different periods. In the first period, there are 55 returns per day over 20 days, so 1,100 returns. In the second period, things pick up substantially. The next 10 days have 80 arriving returns per day, for another 800 returns. So, in total, 1,900 returns arrive over the season.

[Radu writes as Farah speaks.]

$$\begin{aligned} \text{Total returns} &= \text{returns from first 20 days} + \text{returns from next 10 days} \\ &= \frac{55 \text{ returns}}{\cancel{\text{day}}} \cdot 20 \cancel{\text{days}} + \frac{80 \text{ returns}}{\cancel{\text{day}}} \cdot 10 \cancel{\text{days}} \\ &= 1,100 \text{ returns} + 800 \text{ returns} \\ &= 1,900 \text{ returns.} \end{aligned}$$

RADU: But this is just how many returns arrive over the entire season.

FARAH: Right, so now we compute how many days it takes to process the returns. Which is the total number of returns divided by how many returns are processed per day. So for returns, there are 1,900 returns and divide by 50 returns per day, which is 38 days of work.

[Radu writes as Farah speaks.]

$$\text{Days to finish all returns} = 1,900 \cancel{\text{returns}} \cdot \frac{\text{day}}{50 \cancel{\text{returns}}} = 38 \text{ days}$$

RADU: Okay, I am with you. Operations people sometimes call this the *makespan*, the total elapsed time needed to finish all of the jobs.

MING: As much as I enjoy calculations—which I don’t—how exactly does this get us *E*? We are trying to get the number of late returns and this is a measure of time?

RADU: Right, this was my confusion at the beginning.

FARAH: But it’s all there. There are 35 days from the early season until the deadline. It takes 38 days to finish all of the jobs. So this means it takes 3 days to finish the late returns.

MING: Oh, so this means there are 6 cookies left at the deadline. Two cookies per day, and it takes you three days to finish them.

CAMILA: Again with the cookies?

RADU: Right, right, this makes sense. Farah is saying something important here. There are three days of late returns. Each day there are 50 returns processed, so ...*[Turns to the board to write.]*

$$\begin{aligned} E = \text{Total late returns} &= \text{Number of days of late returns} \cdot \text{Amount processed each day} \\ &= 3 \cancel{\text{days}} \cdot \frac{50 \cancel{\text{returns}}}{\cancel{\text{day}}} \\ &= 150 \text{ returns} \end{aligned}$$

In fact, in operations, we sometime refer to inventory in terms of time. We might say, “we have 2 hours of french fries left in the back!” This means that at the current pace, we will run out of french fries in two hours.

FARAH: So yes, exactly, $E = 150$ and so the total late filing fee is 50 dollars per 50 returns or \$7,500.

HAOYU: And we can also solve for the amount we have to pay in additional day late fees.

RADU: And how is that?

HAOYU: We are late for 3 days, and each we pay \$10 per day in late fees, so a total of \$30.

FARAH: No, that can’t be right. That would be the penalty for *one* return that’s 3 days late. There are hundreds of returns spilling past the deadline, not one.

HAOYU: Oh right, then \$10 times 3 days times however many are late, *E* right?

RADU: I like Haoyu’s instinct, but it doesn’t quite work because ...

MING: *[Finishing Radu’s sentence]* Because “3 days” isn’t a number that belongs to all the returns. It’s the worst case, the last return out the door. Most of them are late by less.

RADU: Good. So we need to be more careful about this word “late.” The 10 dollars. 10 dollars per *what*?

HAOYU: Per day.

RADU: Is it? Suppose 20 returns are sitting unfinished, and they sit for 1 day. What’s the bill for that day?

HAOYU: 20 returns, 10 dollars each, so 200 dollars.

RADU: And if those same 20 sit for 2 days?

HAOYU: Oh, I see. Then it's 200 for the first day and 200 for the second. So I total of 400 dollars.

RADU: So the 10 dollars is per day, and ...*[Looks at Camila.]*

CAMILA: And per return. It's 10 dollars per return per day. The charge only makes sense once you say both.

[Radu writes on the board.]

$$\frac{\$10}{\text{return} \cdot \text{day}}.$$

RADU: This is the second time we have seen this in today's discussion. Two things multiplying together in the denominator. Returns times days. Where have any of you seen units multiplied like that before? Maybe in a math or physics class?

FARAH: Area. Square centimeters—centimeters times centimeters. When you multiply two lengths you get an area.

RADU: So if this fee wants to be multiplied by something with units of return-days, it is waiting to be multiplied by ...

[A pause. Several students look at the unit on the board.]

CAMILA: An area. Specifically, an area whose two sides are returns and days. We drew exactly that for Milo's dog wash. The inventory buildup diagram—dogs up the side, time along the bottom, and the area underneath was in dog-minutes.

RADU: Return-days instead of dog-minutes. Camila has helped us unlock something important here. Let's draw an inventory build-up diagram for this.

[Radu turns to the board.]

RADU: Let's go slowly. Binesi, what are my two axes?

BINESI: Time along the bottom. And up the side, the number of returns sitting in the office, waiting to be finished.

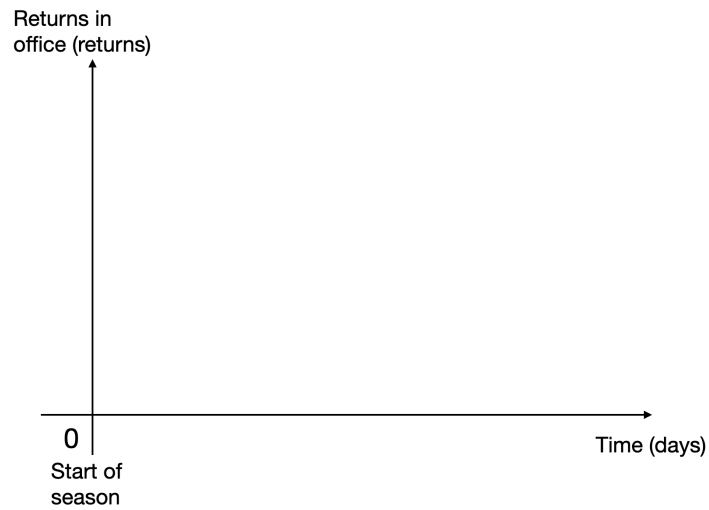
[Radu draws as Binesi speaks.]



RADU: Do you mind if we mark a few significant moments that matter along the bottom, before we draw a single thing going up? Binesi, what should we mark along the bottom?

BINESI: When the interns start and the first returns come in. That is 35 days before the deadline.

RADU: Let's call that Day 0. [*Draws on the board.*]

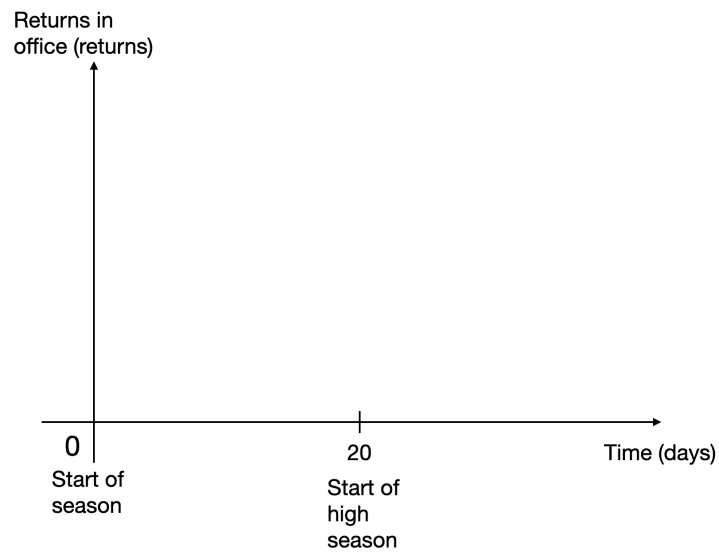


What's next?

BENESI: The pace changes 15 days before the deadline, when arrivals jump from 55 a day to 80.

That's day 20 on your axis.

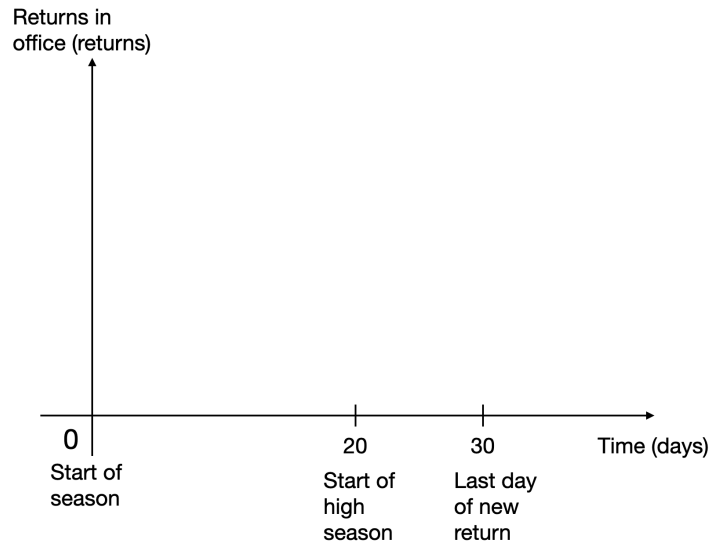
RADU: Good. [*Draws on the board.*]



What's next?

BENESI: 5 days before the deadline there are no new returns. That's day 30.

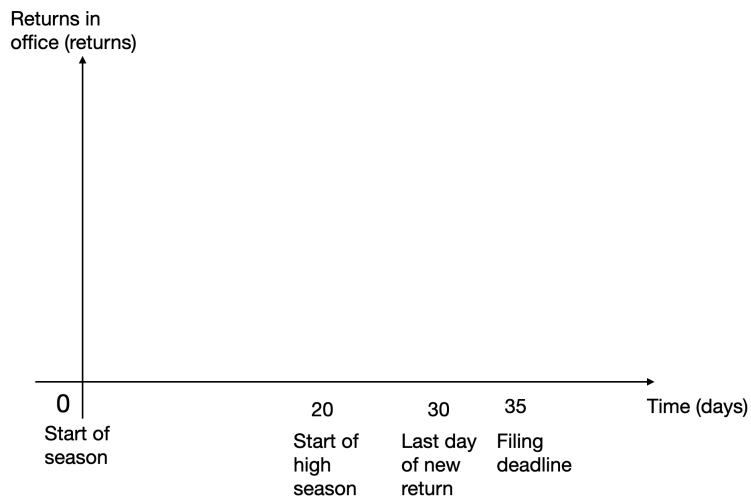
RADU: Perfect. [*Draws on the board.*]



And?

BINESI: The deadline itself. Day 35.

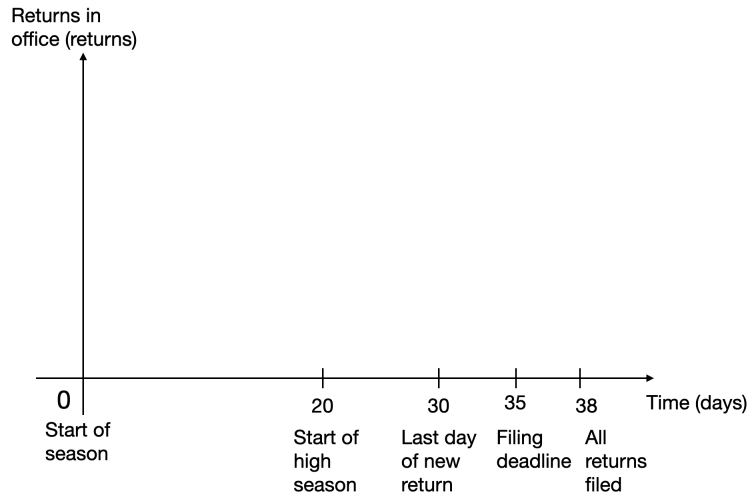
[Radu draws on the board without speaking.]



RADU: Is that the last moment we care about?

BINESI: No. As we calculated before, the work isn't done at the deadline. It's done at day 38, when the last return is finally finished.

[Radu draws on the board without speaking.]



RADU: Now for the vertical. How do we think about inventory going *up*?

BINESI: We saw this the Milo case. Someone arrives and doesn't leave yet. Then a dog joins the line, and the pile ticks up by 1.

RADU: And going down?

BINESI: The opposite. Someone leaves who isn't replaced. A finished return goes out the door, the pile ticks down.

RADU: Here, though, we don't have one dog at a time. Returns are flowing in and out continuously, as rates—55 a day, 80 a day, 50 a day. So how should we think about the pile changing now?

FARAH: By the rates, then. However fast it fills minus however fast it empties.

RADU: By the rates ...*[Stands somewhat puzzled.]*

HAOYU: *[Excitedly]* It's a bathtub.

RADU: A "bathtub"?

HAOYU: The tap pours water in, the drain lets water out. The level rises if the tap beats the drain, and it falls if the drain beats the tap. The returns are the water. Billy's office is the tub. Client returns are the tap, the interns are the drain.

RADU: Very astute, I like it. I take it you don't take showers?

HAOYU: I like a long bath after tea and cookies.

RADU: As one should. *[Turns to the board.]* Let me write Haoyu's bathtub formula down.

$$\text{IN}(t) - \text{OUT}(t) = \Delta I(t)$$

At every time t , the rate in, minus the rate out, is the rate inventory changes. This "delta" here *[Points to the Δ on the board.]* means "change". *[Pauses for a moment to see if students have questions]* Let's use this formula to make sense of what is happening with the inventory of tax returns throughout the tax season.

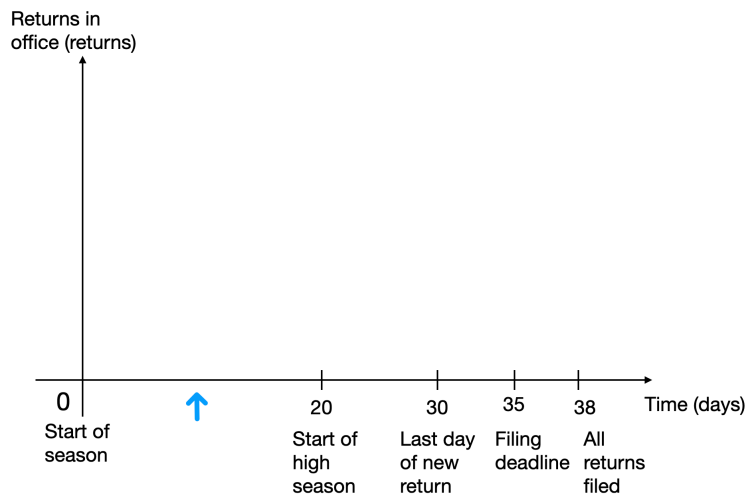
[Radu starts to draw a table.]

Period	IN(t) ($\frac{\text{returns}}{\text{day}}$)	OUT(t) ($\frac{\text{returns}}{\text{day}}$)	$\Delta I(t)$ ($\frac{\text{returns}}{\text{day}}$)
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This table will help us make sense of things. The columns will have our rates in different time periods. The rows are for the different time periods. Why is there going to be more than one row here?

MING: Because the rates are changing over time?

RADU: Exactly. So what is happening in this first time period? [*Points to the board.*]

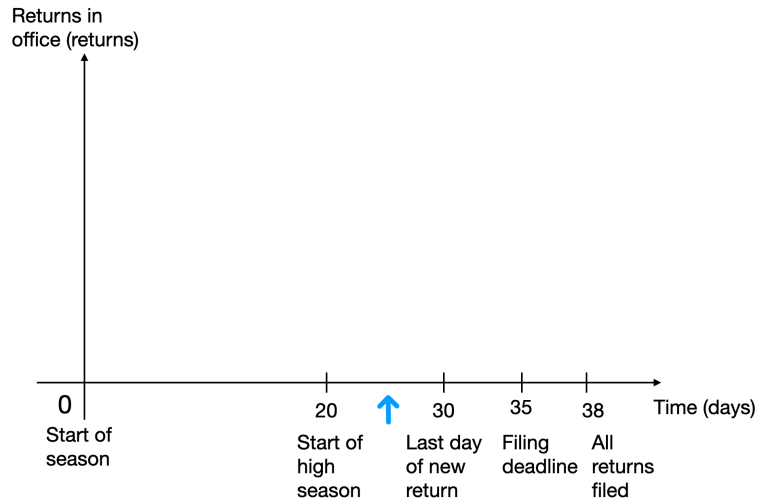


FARAH: The inflow rate is 55 returns a day during this early part of the reason. The outflow rate is 50 returns a day. In that this is the rate of processing returns throughout the whole season. So this means the pile grows by 5 a day.

[*Radu writes as Farah speaks.*]

Period	IN(t) ($\frac{\text{returns}}{\text{day}}$)	OUT(t) ($\frac{\text{returns}}{\text{day}}$)	$\Delta I(t)$ ($\frac{\text{returns}}{\text{day}}$)
Day 0–20	55	50	+5
		50	
		50	

RADU: Camila, this next segment?



CAMILA: This is the high season, so the inflow is 80 returns per day and so the pile grows by 30 returns a day. Much faster.

[Radu writes as Farah speaks.]

Period	IN(t) ($\frac{\text{returns}}{\text{day}}$)	OUT(t) ($\frac{\text{returns}}{\text{day}}$)	$\Delta I(t)$ ($\frac{\text{returns}}{\text{day}}$)
Day 0–20	55	50	+5
Day 20–30	80	50	+30
		50	

RADU: And after they shut the door to news returns on day 30, what happens then?

MING: Nothing's coming in, so the inflow is 0. They're still finishing 50 returns a day, so the pile *shrinks* by 50 returns a day.

[Radu writes as Ming speaks.]

Period	IN(t) ($\frac{\text{returns}}{\text{day}}$)	OUT(t) ($\frac{\text{returns}}{\text{day}}$)	$\Delta I(t)$ ($\frac{\text{returns}}{\text{day}}$)
Day 0–20	55	50	+5
Day 20–30	80	50	+30
Day 30–38	0	50	−50

BINESI: Wait, what happened to the deadline? Why didn't we create a time period that emphasized day 35?

RADU: Good point. Well...

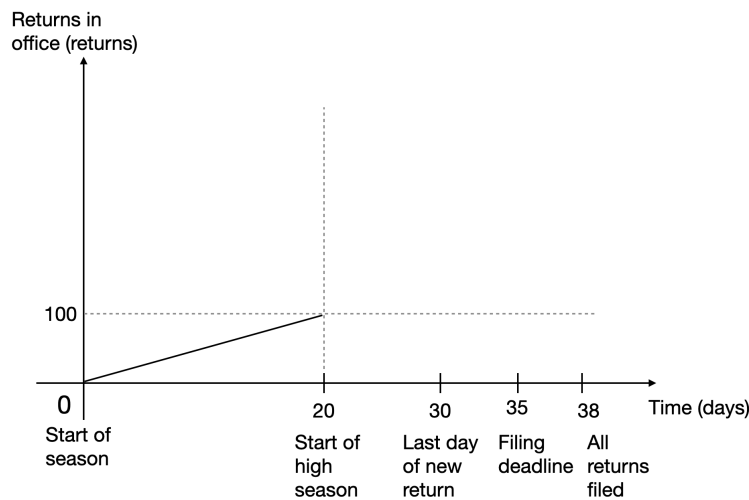
FARAH: The rates don't change on that day, we stopped taking new returns 5 days before the deadline.

BINESI: Right, of course. Stupid question.

RADU: Hardly a stupid question, it completes our story. *[Pauses.]* This table really helps us draw our inventory buildup diagram. Let's start filling in the vertical direction on the figure. How should the diagram look in this first time period?

FARAH: Inventory reaches 100 by day 20. It increases at a rate of 5 returns per day.

RADU: Good, so like a straight line from 0 returns at time 0 to 100 returns at time 20.



MING: *[Interrupts as Radu is still drawing]* Wait. A straight line!? Returns don't come all smooth like that I am sure. I bet you many come in around lunch or end-of-day. The real picture should be at least be a staircase, a little jump every time someone arrives and a little drop every time a return gets finished.

RADU: You're right. That is how it really happens. So why am I drawing a line?

MING: You're asking me? You're the one drawing!

RADU: Yes, because I know you know the answer. Why might a line make sense?

MING: Okay, I see how this is going. It is one of those classes where you turn every question around on us. Congrats.

[The class laughs, including Radu.]

[Pauses for a few seconds] It is a line because there are so many of them?

RADU: So many of what?

MING: If 80 returns arrive in a day and 50 leave, that's 130 little jumps and drops in a single day. If you drew every one, it would like a mess. In the end, a line is a good description of what happens like "on average".

RADU: Which makes sense, doesn't it? After all, the numbers we are given are like averages. That's the whole justification. The arrivals and departures are small and frequent enough that we smooth them into a steady slope. We pretend they trickle in continuously through the day rather than in discrete steps. See Ming, I know you have the answers.

HOAYU: I like it, I'm not too good at drawing little stairs anyway.

RADU: I want to draw one more thing in the picture. Where can I see this number in the picture?

[Points to the table.]

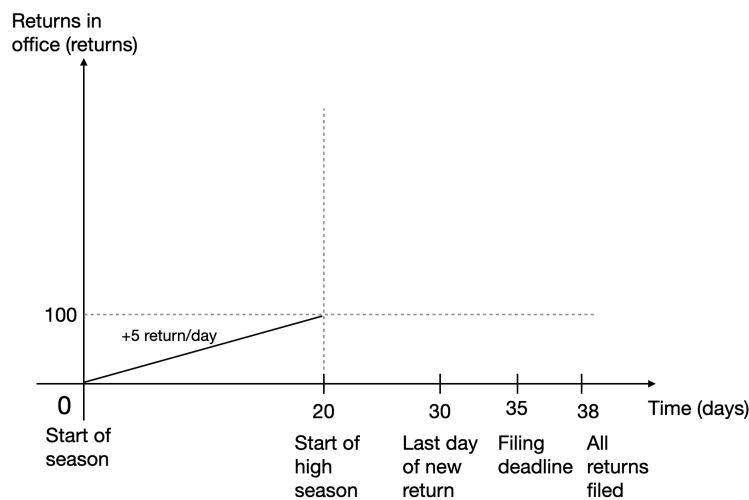
Period	IN(t) ($\frac{\text{returns}}{\text{day}}$)	OUT(t) ($\frac{\text{returns}}{\text{day}}$)	$\Delta I(t)$ ($\frac{\text{returns}}{\text{day}}$)
Day 0–20	55	50	+5✓
Day 20–30	80	50	+30
Day 30–38	0	50	–50

FARAH: That’s the slope of the line you’ve drawn.

RADU: You sure it has the right units?

FARAH: Yes, slope is rise over run. The unit of the rise is “returns” and the unit of the slope is “day”. So the unit of returns per day is perfect here.

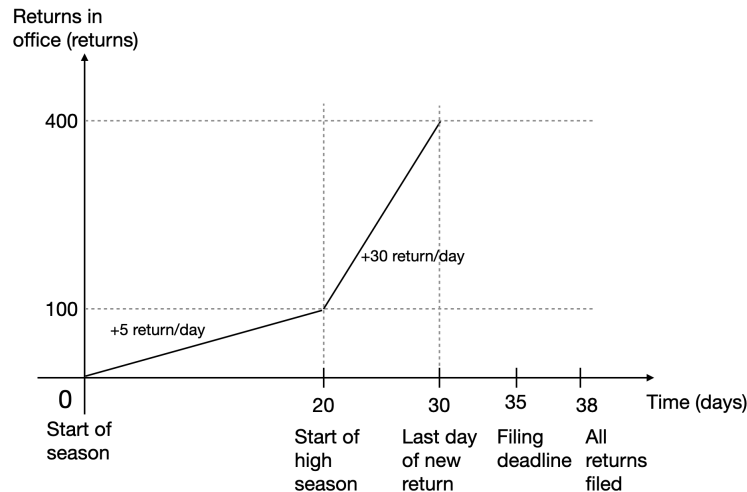
RADU: Good, again, units are important. [*Turns to the board.*]



So, what happens next?

CAMILA: From Day 20 to 30, inventory climbs faster now, 30 a day, starting from 100 it eventually reaches 400 inventory at day 30. That’s an additional 300 returns, 30 returns a day for 10 days is 300 returns.

RADU: [*Drawing the steeper segment up to 400.*] Steeper, because the door’s wide open and the office can’t keep up. Very good. And the slope here is +30.



RADU: And the last segment, after the door closes. Haoyu?

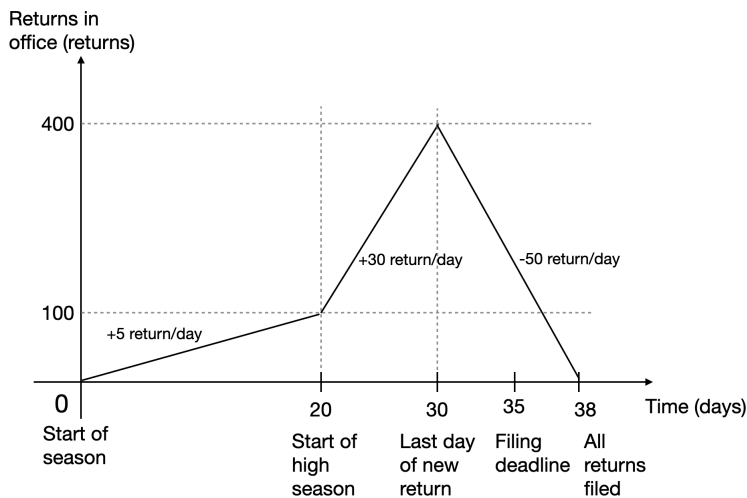
HAOYU: Now it just drains at a rate of 50 returns a day. Starting from 400 until it hits 0.

[Pauses.]

Wait, are we sure it reaches 0 at time 38?

CAMILA: Yes, 400 returns getting consumed at 50 a day. That's 8 days. Starting from day 30, that lands at day 38. Exactly where Farah said the last return goes out.

RADU: [Drawing the line down to 0 at 38.] Down it comes, to nothing.



RADU: Very good, we have drawn an inventory build-up diagram with continuous flows. It complements what we saw in the Milo case, but when the numbers are big like this.

FARAH: It's really nice. I had this same picture in my head all along.

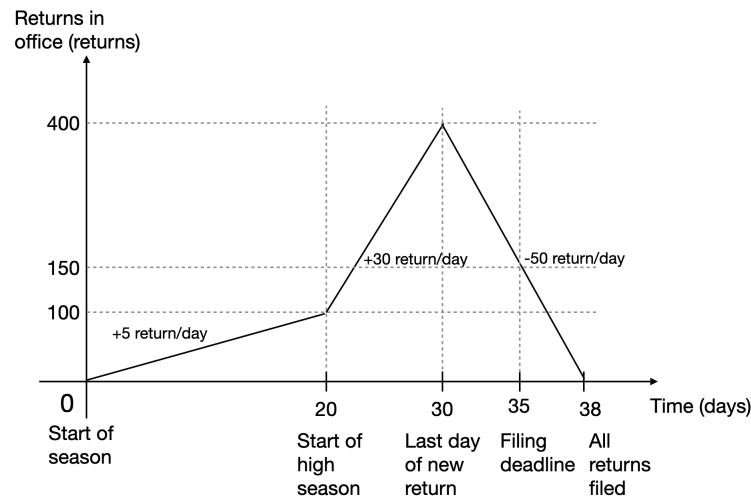
RADU: I am sure you did. Now, somewhere in this picture we can "see" Billy's fees. Where?

BINESI: The fees only starts at the deadline, day 35. Anything finished before then is on time and free. So all returns still in the office above “Day 35” in the graph needs an extension and Billy pays the filing fee.

RADU: And how high is the pile there?

FARAH: This is what we computed before, it’s three days of returns, or 150 returns.

RADU: 150 extension filings at the stroke of the deadline. Perfect. *[Adjusts the diagram]*



This cost Billy \$7,500 dollars. *[Pauses.]* Now, what about these late fees accrued by delays *after* the deadline? Can we see those in the picture too?

BINESI: Before that, I have a question; maybe it seems like a minor point, but I just want to be clear about it.

RADU: No question too minor.

BINESI: Thanks for your patience. So we started on Day 0, right? *[Pointing to the left-most part of the inventory buildup diagram.]*

RADU: Yes.

BINESI: So then Day 1 in the figure means the end of the first day, right? We have already done one whole day of work by the time we get to Day 1.

RADU: Good point, so the start of the first day is at the line for Day 0 and the end of the first day is at the line for Day 1 in the figure.

BINESI: Good, that clarifies it for me. I was wondering if the line above Day 35 was the beginning of the deadline day or the end of it. This confirms that it is the end of the deadline day, and at the end of the day, all of those returns require an extension.

RADU: Excellent point, I overlooked that myself. I started with Day 0 instead of Day 1 largely out of habit, and now I guess we found the reason. *[Pauses.]* So, then let me ask again, can we see these additional penalty fees for delays past the deadline?

FARAH: Aha, I see it clearly now. We should also look at the heights at Day 36, Day 37, and Day 38. Those give counts of returns, 12, and 3 full days late.

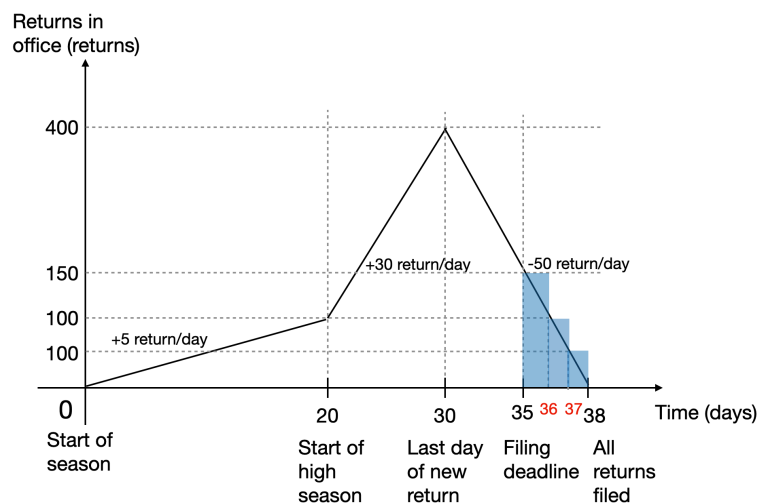
RADU: Like this? *[Turns again to the board.]* These have height 150, 100, and 50, right?

FARAH: Right. I guess usually the policy is that for the returns filed on Day 36, they are counted as 1 day late, even though they may be submitted at the noon time on Day 36. And similarly for returns filed on Day 37.

RADU: So how does this help us?

FARAH: On the beginning of day 36, there are 150 files on Billy's hand, even though he can process 50 that day, they still pass the deadline, and he pays 10 dollars for each one. So 150 times 10 dollars, that's 1500 in total. On day 37, there are 100 files on Billy's hand, and similarly it's 1000 dollar to pay. On day 38, there are 50 files left, and 500 to pay.

[Radu writes as Farah talks.]



RADU: So we had a \$7,500 payout before, and now these additional fees, so we have [Writes on the board.]

$$\text{Total payout} = \$7,500 + \$1,500 + \$1,000 + \$500 = \$10,500$$

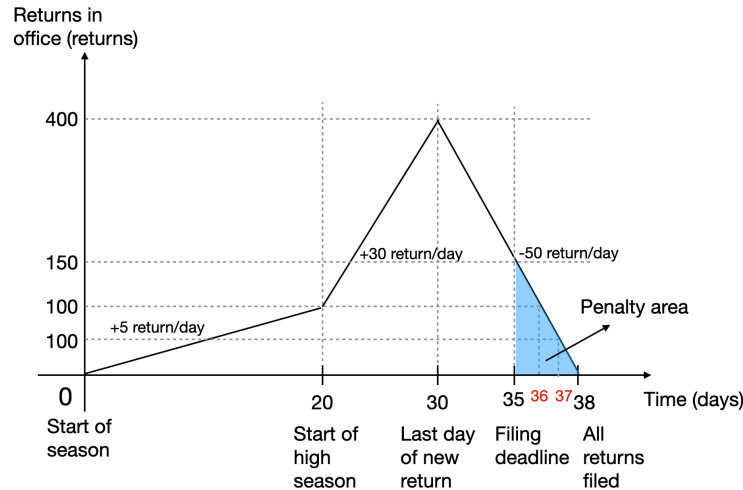
Very good to get this far. [Looks around approvingly.] Now let me change one word in Billy's policy and see what happens. Suppose, instead of charging by the whole day, the fee were *pro-rated*, meaning 10 dollars per return per day, but accruing smoothly by the minute. How would I compute the daily fees then?

FARAH: Then I wouldn't count returns at the end of each day. The returns are leaving smoothly across those last days, so I'd just take the area of that shaded triangle directly and multiply by 10.

RADU: Why is this?

FARAH: This is what we have building to all class. The unit of the fee is 10 dollars per day times return. The unit of *area* in this diagram, as we said before, is day times return. So we should multiply 10 by an area, precisely the area of the triangle under the graph and to the right of the deadline.

[Radu shades in the triangle as Farah speaks.]



RADU: And what is this area?

FARAH: The triangle has a base of 3 days and a height of 150 returns. The area of a triangle is $\frac{1}{2}$ base times height. This gives an area of 225. And so multiply by 10 gives the penalty of 2250 dollars.

$$\begin{aligned}
 \text{Pro-rated time penalty} &= \frac{1}{2} \cdot \text{"Penalty" area} \cdot \frac{\$10}{\text{return} \cdot \text{day}} \\
 &= \frac{1}{2} (3 \text{ days}) \cdot (150 \text{ returns}) \cdot \frac{\$10}{\text{return} \cdot \text{day}} \\
 &= \$2,250
 \end{aligned}$$

RADU: Which means Billy's total payout in the pro-rated case is ...

CAMILA: \$7,500 plus \$2,250, so \$9,750.

MING: The pro-rated has him paying out less?

FARAH: Yes, I think the reason is that under this current policy, he need to pay the full 10 dollars for a return even though the return is not one full day late. But under the pro-rated policy, for example, if he filed a return at the noon on Day 36, he only pays 5 dollars instead of 10 dollars.

RADU: While we have this picture, let me steal two more numbers from it, almost for free. Across the whole season, the full 38 days, what is the *average* number of returns sitting in Billy's office at any given moment?

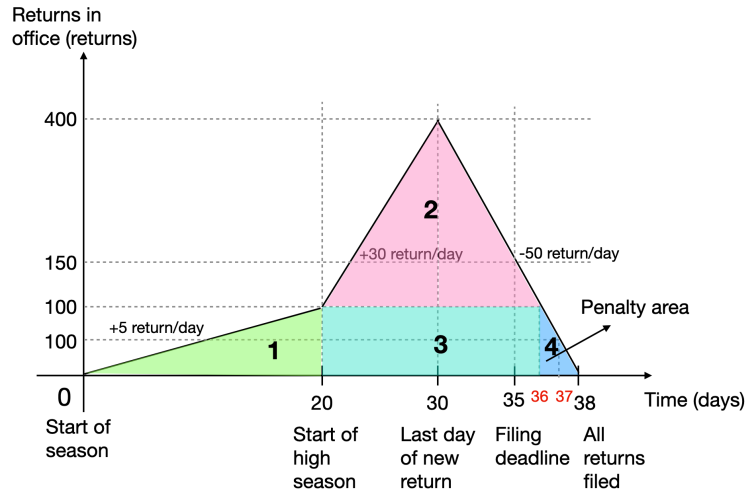
MING: Average height of the whole "mountain" [*Pointing to the shape of the inventory buildup diagram, up from zero and then down to zero.*] And we know how to get an average height—it's the area under the whole thing divided by the width. You put that in an email a few classes back.

RADU: I did, didn't I? You really do read my emails. Remind everyone why area-over-width is an average height.

MING: Because you're asking: what rectangle, sitting on the same base, would hold the same amount as the "mountain"? Its height is the area divided by the base. That flat height is the average.

RADU: So find me the whole area. Not just this final triangle, but the entire “mountain”. Anyone?

CAMILA: I only remember how to do areas of rectangles and triangles: the area of a square is base times height, and the area of triangle is half base times height. I would carve the area up accordingly. *[Stands up and takes the chalk from Radu.]*



Applying the formulas for area we get: the area of (1) is $1/2 \cdot 20 \cdot 100$, the area of (2) is $1/2 \cdot 16 \cdot 300$, and area of (3) is $16 \cdot 100$ and the area of (4) is $1/2 \cdot 2 \cdot 100$. That yields a total of 5,100 day-returns.

RADU: Over a base of?

CAMILA: 38 days. So the average is 5,100 over 38 ...about 134 returns.

BINESI: That seems like a lot just sitting there on average.

RADU: Indeed it does. Now the second free number. *[To Ming.]* How long does the average return spend in the office—from the day someone drops off their shoebox of receipts to the day their return is filed?

MING: You’re going to make me use Little’s Law, aren’t you?

RADU: Am I?

MING: Inventory equals throughput times flow time. We have the average inventory, 134 returns. The throughput is 50 returns a day—they finish 50 returns a day the whole time, and over the season 1,900 returns go in and the same 1,900 come out, so it is a balanced system over the season. So the flow time is inventory over throughput; which is, 134 over 50 ... something between 2 and 3 days.

[Radu writes on the board while Ming speaks]

$$I = RT \Rightarrow T = \frac{I}{R} = \frac{134 \text{ returns}}{\frac{50 \text{ returns}}{\text{day}}} = 2.68 \text{ days.}$$

RADU: That comes to 2.68 days on average. *[Pauses.]* Any final questions about the case?

[Student pause to think about the day’s discussion.]

BINESI: Can I make like a “big idea” comment?

RADU: Always.

BINESI: I’ve been stuck on the policy itself. Billy is *paying* people for being late. If I’m his client and I owe a lot of tax—the government charges its own interest and penalties on what I owe if I’m late. Also, being late can cause more scrutiny or demerits by the tax collector. Billy’s 80-dollar guarantee doesn’t come close to covering that for me. I’m not sure I’d hand my return to a man who’s this comfortable missing the deadline.

MING: But would the people who owe a lot really be the ones showing up on day 30? If I owed the government a fortune, I’d want it filed early, not thrown on top of a pile of 150.

BINESI: Maybe.

MING: The last-minute crowd is probably people who owe little or are getting a refund. For them, 80 dollars is found money.

CAMILA: Which is exactly the problem. If the penalty is pure upside for the latecomers, you’ve built an incentive to come late. Why would I drop my return off on day 25 when I could stroll in on day 30, guarantee myself a spot in the late pile, and collect Billy’s money?

HAOYU: [*Grinning.*] You’d be fishing for the penalty. I love it.

RADU: [*Raising a hand.*] Before the three of you redesign Billy’s business so he pays all of you 80 dollars apiece, let’s end the discussion here.

HAOYU: Well, I do need some way to pay for all the tea and cookies I bring to class.

[*Everyone laughs and heads out of the classroom.*]

Email

Dear students,

A long and lively class, and I left it feeling that we got more out of one picture than I had any right to expect. So let me say what I think happened. We drew one shape, a mountain of returns rising and falling over 38 days, and then we read that same shape four different ways. Its *height* at any moment was the inventory in Billy’s office. A particular *area* inside it was the penalty he owed. The *whole* area, divided by its width, was the average inventory. And Little’s Law turned that average into the *time* a return spends on his desk. Four questions, one figure. If you remember nothing else, remember that a well-drawn inventory buildup diagram is a powerful tool.

Now let me develop a few threads we brushed against but did not have time to follow.

Inventory build-up diagrams and the meaning of area

At the center of today’s class was the *inventory build-up diagram*. This is a two-dimensional graph, and I want to emphasize that understanding these diagrams is not optional in this course—they are one of our main ways of thinking. Every inventory build-up diagram has at least these three essential components:

- The **horizontal axis** represents *time*, measured in units such as minutes, hours, or days.
- The **vertical axis** represents *inventory*, meaning a count of items in the system (dogs, jobs, customers, books, assignments).

- The **area under the curve** represents accumulated inventory over time.

It is this third component—the area—that often feels the least intuitive at first, so let us slow down and make it concrete. If inventory is measured in *jobs* and time is measured in *hours*, then the area under an inventory curve has units

$$\text{jobs} \cdot \text{hours}.$$

often shorted to “job-hours”. This unit may look unfamiliar, but you have encountered it many times in everyday life.

Consider a simple example involving library book late fees. Suppose a library charges 25 cents per book per day for late returns. Written in units, this charge rate is

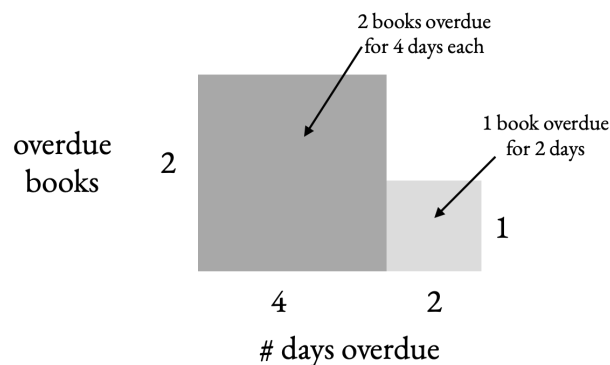
$$\frac{25 \text{ cents}}{\text{book} \cdot \text{day}}.$$

Notice that we have the units “book-days” on the bottom here. It is colloquial to say the charge is 25 cents per day, but this is not accurate. You are not charged 25 cents per day every day you have a book that is late. You are charged 25 cents per day *per book*.

Now imagine that over the course of a week, you have two books overdue for four days, and then one book overdue for an additional two days. How would calculate the late fee? Probably something like:

$$(2 \text{ books} \times 4 \text{ days} + 1 \text{ book} \times 2 \text{ days}) \times \frac{25 \text{ cents}}{\text{book} \cdot \text{day}} = 10 \text{ book-days} \times \frac{25 \text{ cents}}{\text{book-day}} = 250 \text{ cents}.$$

But did you notice that have just done an area calculation here? One can draw an inventory build-up diagram where the vertical axis counts late books and the horizontal axis measures days:



The area here is

$$(2 \text{ books} \times 4 \text{ days}) + (1 \text{ book} \times 2 \text{ days}) = 10 \text{ book} \cdot \text{days}.$$

In other words, the above calculation is nothing other than:

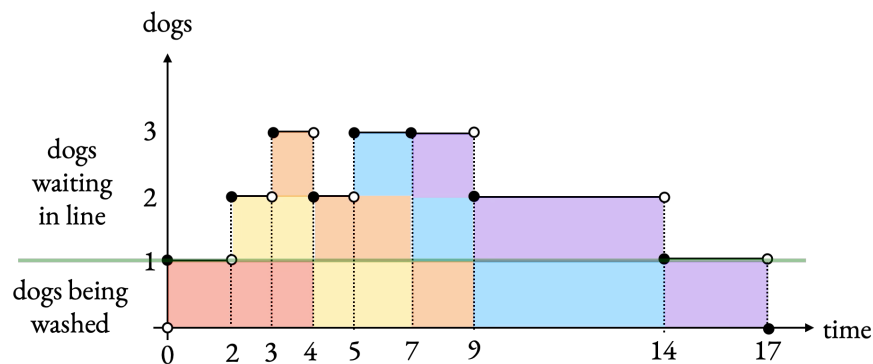
$$\text{Area under the inventory buildup diagram} \times \frac{25 \text{ cents}}{\text{book} \cdot \text{day}} = 10 \text{ book-days} \times \frac{25 \text{ cents}}{\text{book-day}} = 250 \text{ cents}$$

Let's solidify this by returning for a moment to Milo's dog wash business from a few classes back. Suppose his neighbors at the farmer's market get so upset and band together to start taxing him 10 cents per minute per dog waiting in his line. How much would his total tax bill be for the first 17 minutes of his day?

Well, the tax has exactly the structure of the library late fees:

$$\frac{10 \text{ cents}}{\text{dog} \cdot \text{minute}}.$$

It wants to be multiplied by an area. So we can calculate the area under the inventory buildup diagram, but being careful not to include dogs inside the van. The relevant figure in the Milo dog-washing case is:



The relevant area is the area under the curve but above the green line, which you can check is 17 dog-minutes.

Haoyu's bathtub is a conservation law. When Haoyu said the office was a tub, the returns were water, the clients were the tap, and the interns were the drain, he was not merely reaching for a homely picture. He had stated a law. The rule I wrote on the board,

$$\text{IN}(t) - \text{OUT}(t) = \Delta I(t),$$

is the same piece of bookkeeping that physicists call a *continuity equation* or a *mass balance*: the rate at which a quantity accumulates in a region equals the rate flowing in minus the rate flowing out. Water in a tub obeys it. So do the molecules in a stretch of pipe, the charge in a wire, the money in a bank account, the carbon in the atmosphere. Nothing appears or vanishes inside the box; the level changes only because of what crosses the boundary. It is one of the most reliable ideas in all of science, and you now own a version of it.

Why I was allowed to draw a smooth line. Ming was right to object. Returns do not arrive in a smooth ramp; they arrive one at a time, a person through the door, an email with a soft chime. The true inventory curve is a staircase with a hundred and thirty tiny steps a day, not the clean line I drew. What I did, and what we will do again and again, is replace that jagged reality with a smooth flow at a steady rate. Operations researchers call this a *fluid* or *continuous approximation*: treat the discrete trickle as if it were a liquid pouring at 55 a day, or 80, or draining at 50. The approximation is good precisely under

the conditions Ming identified: when the individual units are small relative to the pile and the events are frequent. Fifty returns finished in a day, against a backlog of hundreds, is getting pretty close to water.

The bathtub is harder than it looks, and that is not your fault. I want to reassure those of you who found the buildup diagram slow going. The difficulty is real and well documented. Some years ago, John Sterman and Linda Booth Sweeney ran a now-famous experiment at MIT: they handed business-school graduate students who are quantitatively trained a picture of a bathtub together with graphs of the water flowing in and the water flowing out, and asked them simply to sketch how the water level would change over time. Fewer than half got it right [6]. Later work found the same failure in even simpler settings, and gave the phenomenon a name: *stock-flow failure*, which refers to our chronic inability to reason from the rates of flow to the level of the stock. So if it took real effort to see that the pile keeps *growing* through the whole first thirty days, even as the office runs flat out at 50 a day, you are in excellent company. The seductive error is to think that because the office is always busy, the backlog must be shrinking. It is not. The tap beats the drain, and the tub fills, no matter how hard the drain works. That single confusion, more than any arithmetic, is what makes Billy’s problem worth our attention.

Units as a divining rod. Twice today the units did our thinking for us. I want to make that habit explicit, because it is one of the most powerful and underused tools you have. I spoke about this in an earlier email, but now we see the idea in full force. Early on in our discussion, my equation for the office’s rate had units that would not cancel: I had “interns” multiplied by returns per day, and no way to make the “interns” disappear, until Camila pointed out that each intern’s rate was really 10 returns per day *per intern*, at which point the interns cancelled and left a clean rate of returns per day. The second time, the penalty’s units, dollars per return per day, told us it was waiting to be multiplied by an area measured in return-days, which is exactly what the buildup diagram supplies. In both cases we reasoned about the world by reasoning about the units, before touching a single number.

This is not a classroom trick; it is a principle with a serious pedigree. Joseph Fourier, in his 1822 treatise on the theory of heat, insisted that every equation describing a physical quantity must be *dimensionally homogeneous*, i.e., the units on both sides must match, or the equation is simply wrong, whatever its numbers say [4]. Nearly a century later the physicist Edgar Buckingham turned this insistence into a formal method for deducing the shape of physical laws from their dimensions alone [3]. My own instinct in class—“if my units are not working, I am missing something”—is a small, domesticated version of Fourier’s law. When the units refuse to cancel, do not force the arithmetic. Stop, and ask what you have misunderstood about the quantities themselves.

Where the peak comes from, part one: the price. We took Billy’s arrival pattern as given: 55 a day, then a surge to 80, while an economist would immediately ask why the surge is allowed to exist at all. The whole problem is that demand is not level: the office could comfortably handle 50 a day forever, and it is only the crush above 50 that builds the backlog and buys the penalties. There is a classic response to peaked demand, worked out for electricity and telephone networks in the 1950s by Marcel Boiteux in France and Peter Steiner in the United States: charge more at the peak [2, 5]. If capacity is scarce precisely when everyone wants it, then the price during that window should reflect that scarcity. Billy does the opposite. He charges one flat fee whenever you walk in, and then *pays* the peak-season latecomers on top of it. A peak-load logic would have him discount the quiet early weeks and charge a premium for a return dropped off in the final ten days, nudging the flexible clients to arrive when the interns are idle. He would

be using price to flatten the mountain rather than penalty to pay for it.

Where the peak comes from, part two: the deadline itself. But price is not the whole story, because the peak is not only about money. It is about human frailty in the face of a deadline. People procrastinate, they know they procrastinate, and they sometimes take costly steps to protect themselves from it. Dan Ariely and Klaus Wertenbroch ran a lovely study on exactly this: they let students set their own deadlines for three papers, with real grade penalties for lateness, and found that students voluntarily imposed early, binding deadlines on themselves—accepting a risk they could have avoided—because they understood that a wall of “all due at the end” would sink them [1]. Now look again at Billy’s hard cutoff, five days out, posted on the door. We treated it as an operational fact, a rate that drops to zero on day 30. But it is also a commitment device—Billy imposing on his clients the very discipline the Ariely and Wertenbroch students imposed on themselves. Without that door, the 80-a-day surge would run right up to the deadline and beyond, and the penalty bill would be far worse.

There is something fitting in all of this. Billy made a warm, generous promise, *your deadline is my deadline*, and never sat down to see what it cost. When we finally did, the figure was \$10,500 a spring, a number he had been paying for years without seeing, hidden inside a mountain no one had drawn. That is, in the end, what this course is for: to draw the mountain, and see what it tells us.

See you next class,

Radu

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- [4] Jean Baptiste Joseph Fourier. *Théorie analytique de la chaleur*. Firmin Didot, Père et Fils, Paris, 1822.
- [5] Peter O. Steiner. Peak loads and efficient pricing. *Quarterly Journal of Economics*, 71(4):585–610, 1957. doi: 10.2307/1885712.
- [6] Linda Booth Sweeney and John D Sterman. Bathtub dynamics: initial results of a systems thinking inventory. *System Dynamics Review: The Journal of the System Dynamics Society*, 16(4):249–286, 2000.

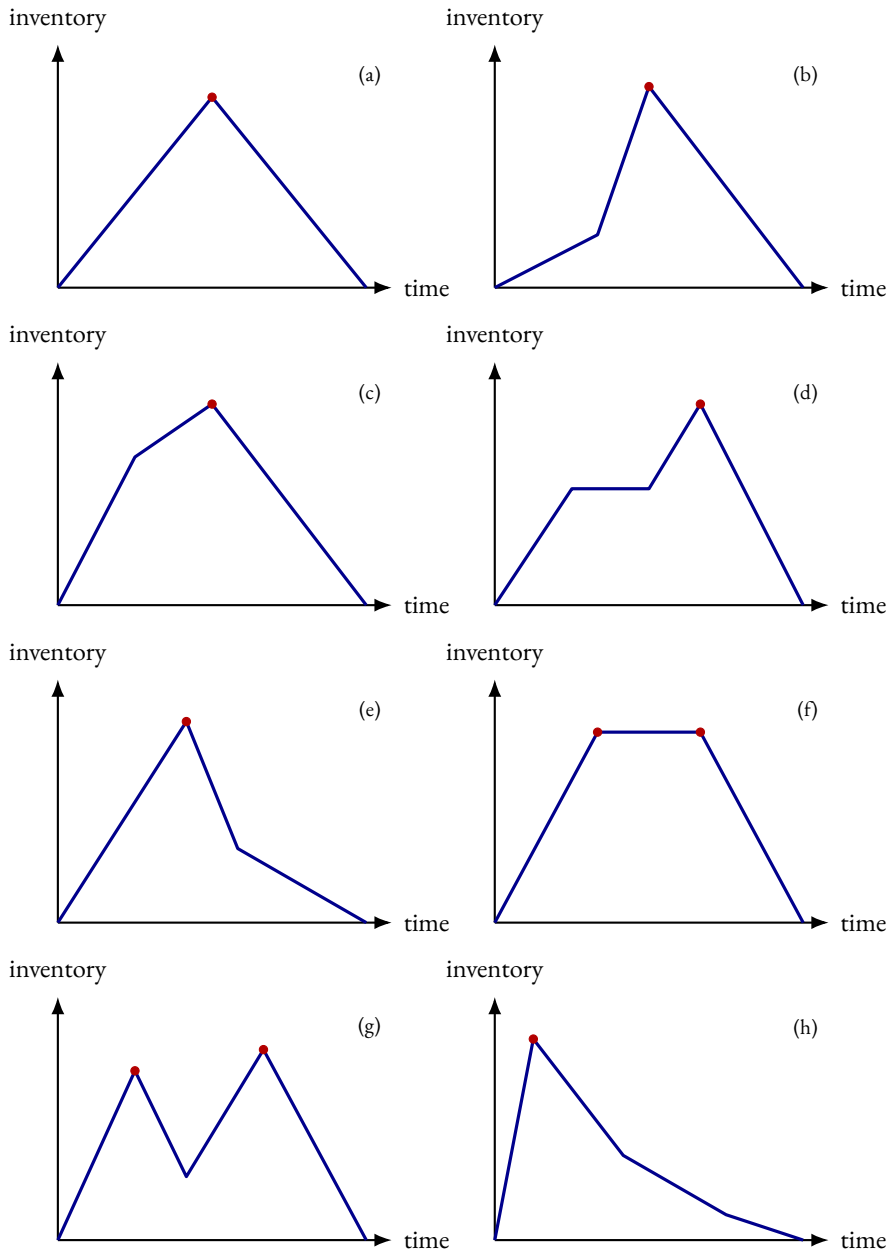
Extensions

[Posted as homework on the course's website for this class session.]

1. Billy's promise that \$50 plus \$10 per return per day late costs him about \$10,500 a season, and the class noticed it could potentially reward latecomers. Design a *different* guarantee that better serves Billy's actual goal (signaling reliability to clients) without paying people to procrastinate. State your policy precisely, and describe how you might estimate its cost using an inventory buildup diagram, and say in words how you think each type of client—the early filer, the anxious procrastinator, the person who owes a fortune—would respond.
2. Billy stops taking new returns 5 days before the deadline. Suppose he moved the cutoff earlier (say 8 days out) or later (say 2 days out). For each, redraw the inventory buildup diagram, recompute the extension count and total penalty, and describe the human trade-off. Where would *you* set the cutoff, and what would you need to know about Billy's clients and margins to defend the choice?
3. Billy considers hiring one more intern, raising capacity from 50 to 60 returns per day. Redraw the inventory buildup diagram, and recompute the extension count, the total penalty, the average inventory, and the average flow time. Then reason about whether the intern pays for themselves: roughly what would an intern have to cost, per season, for this to be worth it purely on penalty savings? What other factors might your answer be missing?
4. Each expression below is something a manager might write about Billy's office. For each, work out the units, say whether the result is a count, a rate, a time, or an area, and state in one sentence what real-world quantity it represents, or explain why it is nonsense:
 - (a) $1,900 \text{ returns} \div 50 \frac{\text{returns}}{\text{day}}$
 - (b) $150 \text{ returns} \times \$10 / (\text{return} \cdot \text{day})$
 - (c) $5,100 \text{ day-returns} \div 1,900 \text{ returns}$
 - (d) $50 \frac{\text{returns}}{\text{day}} \times \$10 / (\text{return} \cdot \text{day})$

(The last one is a trap. What has gone wrong, and what would you have to multiply or divide it by to rescue it into something meaningful?)
5. The chapter dwells on strange multiplied units: return-days, dog-minutes, book-days. Invent a plausible business or everyday situation whose natural "cost" has units of (something) $\times$ (something) in the denominator. Describe the situation, write the unit, draw a small inventory buildup diagram for it, and show how an area calculation gives the total cost.
6. In class, we smoothed 130 daily arrivals and departures into a clean line, justified because the steps are tiny relative to the pile. Construct a version of Billy's business where that smoothing would be *dangerous*, where treating the jagged reality as a smooth flow would give a materially wrong answer. What changed to break the approximation, and how would you know, from the numbers alone, that you were no longer allowed to draw the line?

7. Suppose Marchetti & Singh splits into two smaller offices across town, each hiring interns and each facing half the arrivals (so about 27–28 returns a day early, 40 a day late). Assuming each office has capacity 25, work out the total penalty across both offices and compare it to the single combined office's \$10,500. Is splitting better or worse for total penalties? Explain what property of the buildup diagram drives your answer, and whether “pooling” the two offices back together would help.
8. Below is an inventory buildup diagram for a different business over one day (you may sketch any single-peak mountain that rises and falls to zero). Without any words attached, infer a plausible story: what is the flow unit, what causes the rise, what causes the fall, what is the busiest moment, and what would the area under the curve *cost* someone if there were a holding charge? Invent the business and defend your reading of every feature of the shape.



Reflections

[Posted as a reflection exercise on the course's website for students to write about in a journal due at the end of semester.]

1. Farah wanted to “just compute the answer and show it works,” and Radu made her slow down and justify where she was headed. When you already know how something turns out, does explaining it to someone else feel like a gift or a chore? What does your honest answer say about you?

2. Camila reached for cookies to explain a division, and the class never quite let the cookies go. Think of an analogy you lean on to make sense of something hard. Does it still illuminate the thing? Or have you started mistaking the analogy for the thing itself?
3. Radu described inventory in units of time: “we have two hours of fries left.” Try describing something you’re low on right now not as an amount but as a length of time until you run out. Did that reframing change how urgent it feels?
4. Binesi apologized for a “stupid question” about whether a mark meant the start or the end of the day, and it turned out to pin down the whole calculation. When did a question you were embarrassed to ask turn out to be the one that mattered? What stops you from asking it sooner?
5. Radu wouldn’t move on while his units refused to cancel: “if my units aren’t working, I am missing something.” Is there a small inconsistency you’re currently stepping over? Maybe a number that doesn’t quite add up, or a story that doesn’t quite cohere, because chasing it feels like a waste of time?
6. Billy built his whole reputation on a warm promise he had never actually priced. What have you promised—to a friend, a family member, an employer—without ever sitting down to reckon its real cost to you? Are you still paying it?
7. The class realized Billy’s generous guarantee might be pure profit for exactly the people who least deserve it. Have you ever offered something kind and open-handed, only to watch it get taken advantage of? Did you stop offering it, narrow it, or keep giving anyway? How do you feel about that choice now?
8. The average number of returns just *sitting* in Billy’s office was 134, a pile no one had likely ever drawn or noticed. What is the equivalent pile in your own life or work: the accumulation you live beside every day without ever measuring? What do you think you’d find if you drew it?