THE SLATER CONUNDRUM: DUALITY AND PRICING IN INFINITE DIMENSIONAL OPTIMIZATION

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Abstract. Duality theory is pervasive in finite dimensional optimization. There is growing interest in solving infinite-dimensional optimization problems and hence a corresponding interest in duality theory in infinite dimensions. Unfortunately, many of the intuitions and interpretations common to finite dimensions do not extend to infinite dimensions. In finite dimensions, a dual solution is represented by a vector of "dual prices" that index the primal constraints and have a natural economic interpretation. In infinite dimensions, we show that this simple dual structure, and its associated economic interpretation, may fail to hold for a broad class of problems with constraint vector spaces that are Riesz spaces (ordered vector spaces with a lattice structure) that are either σ-order complete or satisfy the projection property. In these spaces we show that the existence of interior points required by common constraint qualifications for zero duality gap (such as Slater's condition) imply the existence of *singular* dual solutions that are difficult to find and interpret. We call this phenomenon the Slater conundrum: interior points ensure zero duality gap (a desirable property), but interior points also imply the existence of singular dual solutions (an undesirable property). Riesz spaces are the most parsimonious vector-space structure sufficient to characterize the Slater conundrum. Finally, we provide sufficient conditions that "resolve" the Slater conundrum; that is, guarantee that in every solvable dual there exists an optimal dual solution that is not singular.

Key words. Infinite dimensional optimization, duality theory, Riesz spaces

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1. Introduction.

Duality is one of the most useful tools for modeling and solving optimization problems. Properties of the dual problem are used to characterize the structure of optimal solutions and design algorithms. The dual may be easier to solve than the primal and there exist well-known sufficient conditions, such as Slater's constraint qualification, that imply *zero duality gap* between the optimal values of the primal and dual.

Many real world applications are naturally modeled in infinite dimensions, with examples in revenue management (Gallego and van Ryzin, [15]), procurement (Manelli and Vincent, [25]) inventory management (Adelman and Klabjan [1]), territory division and transportation (Carlsson [10]) In each case, the authors apply properties of the dual to simplify the problem and develop structure for optimal solutions.

The focus of our research is on the connection in infinite dimensional convex optimization between constraint qualifications for zero duality gap and the existence of optimal dual solutions that are easily characterized and have a meaningful economic interpretation. A precise overview of our main results is found in Section 1.2.

To motivate our results, we begin with concrete examples. These examples illustrate how the intuitions and interpretations common in finite dimensions do not necessarily hold in infinite dimensions. Duality concepts that do extend to infinite dimensional problems are more subtle and difficult to apply in practice. This development is meant to be accessible to readers with little or no background in functional analytic approaches to convex optimization.

1.1. Motivation.

In finite-dimensional convex optimization, conditions for zero duality gap (such as Slater's constraint qualification) are well understood. Moreover, there is a standard form and economic interpretation of the dual. Researchers in operations research and economics typically define a vector of dual prices that index the (finitely many) constraints. Each price is interpreted as the marginal value of the constraining resource of the corresponding constraint. A real vector of dual prices in finite dimensional convex optimization is a convenient representation of a linear functional defined over the constraint space, termed a *dual functional*. It is this notion of dual functionals, possibly no longer representable as a real vector, that extends to infinite dimensional problems and allows us to build a duality theory for convex optimization problems over arbitrary ordered vector spaces. In the economics literature these dual functionals are called *prices* (see, for instance [12]). This is valid terminology since dual funtionals "price" constraints by mapping vectors in the constraint space to the real numbers. When there is a zero duality gap between an optimization problem and its Lagrangian dual, this pricing interpretation gives significant insight into the structure of the primal optimization problem. However, in infinite dimensions the connection between conditions for zero duality gap and interpretations of the dual are more complex. The following examples illustrate this complexity.

Example 1.1. Consider the finite dimensional linear program

$$\begin{aligned} &\min x_1 \\ &x_1 \geq -1, \\ &-x_2 \geq 0, \\ &x_1 - \frac{1}{i}x_2 \geq 0, \quad i = 3, 4, \dots, 10. \end{aligned} \tag{1.1}$$

The vector spaces used in this problem are easily characterized. The *primal variable space* $X = \mathbb{R}^2$ contains the feasible region and the *constraint space* $Y = \mathbb{R}^{10}$ contains the problem data for the primal constraints and is ordered by the cone $\mathbb{R}^{10}_+ = \{y \in \mathbb{R}^{10} : y_i \geq 0, \, i = 1, \dots, 10\}$. The dual feasible region is a subset of the vector space Y' of linear functionals that map Y into $\mathbb{R}$ (see Luenberger [23]). The space Y' is called the *algebraic dual* of Y and elements of Y' are called *dual functionals*. The dual constraints of (1.1) are

$$\psi((1, 0, 1, \dots, 1)) = 1, \tag{1.2}$$

$$\psi((0, -1, -1/3, \dots, -1/10)) = 0, \tag{1.3}$$

$$\psi \in (\mathbb{R}^{10})'_+. \tag{1.4}$$

Consider the dual constraint (1.2) and let $e_i \in \mathbb{R}^{10}$ be the vector that equals 1 in the i^{th} component and 0 otherwise. Since every dual functional ψ is linear, we write (1.2) as

$$\begin{aligned} \psi((1, 0, 1, \dots, 1)) &= \psi(1 \cdot e_1 + 0 \cdot e_2 + \cdots + 1 \cdot e_{10}) \\ &= \psi(1 \cdot e_1) + \psi(0 \cdot e_2) + \cdots + \psi(1 \cdot e_{10}) \\ &= 1 \cdot \psi(e_1) + 0 \cdot \psi(e_2) + \cdots + 1 \cdot \psi(e_{10}). \end{aligned}$$

Therefore, ψ can be represented by a real vector $\psi_i := \psi(e_i)$ for $i = 1, \dots, 10$. Using this notation, the dual of (1.1) is

$$\begin{aligned} &\max -\psi_1 \\ \psi_1 + \psi_3 + \psi_4 + \cdots + \psi_{10} &= 1, \\ -\psi_2 - (\psi_3/3) - (\psi_4/4) - \cdots - (\psi_{10}/10) &= 0, \\ \psi_i &\geq 0 \quad i = 1, 2, \dots 10. \end{aligned} \tag{1.5}$$

Representing the dual functional ψ as the real vector $(\psi_1, \dots, \psi_{10})$ is standard practice in finite dimensional optimization. Problem (1.5) is a finite dimensional linear program with a simple structure: the constraint matrix of the dual program is the *transpose* of the constraint matrix of the primal. This property may not hold in infinite dimensional problems, as seen in Example 1.2 below.

If the primal is feasible and bounded, then there is an optimal primal and dual solution with zero duality gap. An optimal primal solution is $(x_1^*, x_2^*) = (-1, -10)$ and an optimal dual solution is $(\psi_1^*, \ldots, \psi_{10}^*) = (1, 0, \ldots, 0)$, each with an objective value of -1.

The representation of the dual functional ψ as the real vector $(\psi_1, \ldots, \psi_{10})$ is convenient for interpreting the dual. The optimal value of ψ_i is the increase in the primal objective value resulting from one unit increase in the right-hand side of the i^{th} primal constraint. Therefore, when the value of the primal and dual are equal, optimal dual prices are the marginal prices a decision maker is willing to pay to relax each primal constraint. This relationship between dual functionals and the pricing of constraints is a key modeling feature behind the success of duality theory of finite dimensional optimization. $\triangleleft$

The next example demonstrates that many of the nice properties of finite dimensional optimization in Example 1.1 may fail to hold in infinite dimensions.

Example 1.2 (Karney [21], Example 1). Consider the extension of Example 1.1 to infinitely many constraints but still finitely many variables.

$$
\begin{aligned}
\inf x_1 & \\
x_1 &\geq -1, \\
-x_2 &\geq 0, \\
x_1 - \frac{1}{i}x_2 &\geq 0, \quad i = 3, 4, \ldots.
\end{aligned}
\tag{1.6}
$$

The left-hand-side column vectors $(1, 0, 1, 1, \ldots)$ and $(0, -1, -1/3, -1/4, \ldots)$ and right-hand-side vector $(-1, 0, \ldots)$ belong to many choices of constraint space. This is typical in infinite dimensions where multiple nonisomorphic vector spaces are consistent with the constraint data. By contrast, in finite dimensions all finite dimensional vector spaces of dimension m are isomorphic to $\mathbb{R}^m$.

A natural choice for the constraint space of (1.6) is the vector space $\mathbb{R}^{\mathbb{N}}$ of all real sequences. A naive approach to formulating the dual of (1.6) when the constraint space is $\mathbb{R}^{\mathbb{N}}$ is to mimic the logic of Example 1.1: assign a real number ψ_i to each constraint $i = 1, 2, \ldots$, and then take the transpose of the primal constraint matrix. Assuming this representation is valid, define a dual functional ψ on the constraint space of (1.6) by an infinite sequence $\{\psi_i\}_{i=1}^{\infty}$ where $\psi_i := \psi(e_i)$ with e_i, again having 1 in the i^{th} component and 0 elsewhere. However, this representation of an arbitrary dual functional ψ is not valid unless the following condition holds. Let $\{a_i\}_{i=1}^{\infty}$ be an arbitrary vector in the constraint space. We say a dual functional ψ is *countably additive* if $\psi\left(\{a_i\}_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} a_i\psi_i$. Fortunately, when the constraint space vector space is $\mathbb{R}^{\mathbb{N}}$, all positive dual functionals are countably additive. Indeed, Basu, Martin and Ryan [7] prove that positive dual functionals in the algebraic dual of $\mathbb{R}^{\mathbb{N}}$ can be expressed as positive sequences with finite support; that is, $\psi_i > 0$ for only finitely many $i \in \mathbb{N}$. Clearly, such dual functionals are countably additive.

The dual program derived by taking the transpose of the constraint matrix in (1.6) is

$$
\begin{aligned}
&\sup -\psi_1\\
\psi_1 + \psi_3 + \psi_4 + \cdots &= 1,\\
-\psi_2 - (\psi_3/3) - (\psi_4/4) - \cdots &= 0,\\
\psi_i &\geq 0, \quad i = 1, 2, \ldots\\
\psi_i &> 0, \quad \text{for at most finitely many } i.
\end{aligned} \tag{1.7}
$$

Notice that (1.7) closely resembles its finite dimensional analogue (1.5). The justification for this dual formulation is based on countable additivity and is formalized in Section 3.1 (see (3.5)). Unlike the finite case, a duality gap exists between (1.6) and (1.7). The optimal solution to the primal program (1.6) is $(x_1^*, x_2^*) = (0, 0)$ with a value of 0. Since every feasible solution to the dual program (1.7) is nonnegative, the second constraint implies ψ_i is zero for all $i \in \{2, 3, \ldots\}$. Therefore, the optimal dual solution is $(1, 0, 0, \ldots)$ with a value of -1.

The existence of a duality gap may be surprising to readers familiar with Slater's constraint qualification in finite dimensions. Slater's result states that if there exists a primal solution x such that each constraint is strictly satisfied, then there is a zero duality gap. There are feasible points of (1.6) ($\bar{x} = (1, -1)$ for instance) that strictly satisfy each constraint and the existence of a duality gap appears to contradict Slater's constraint qualification. How can this be? In infinite dimensions, feasible points that strictly satisfy each constraint are not necessarily interior points of the positive cone of the constraint space, the condition required by infinite dimensional versions of Slater's result (see Theorem 2.1). Indeed, the positive cone of the vector space $\mathbb{R}^{\mathbb{N}}$ has an empty interior under every linear topology (Aliprantis and Border [2]).

Alternatively, choose the constraint space of (1.6) to be the vector space ℓ_∞ of bounded sequences. This is a valid constraint space because the columns and the right-hand side of (1.6) lie in ℓ_∞. The resulting dual program is

$$
\begin{aligned}
&\sup \psi(\{-1, 0, \ldots\})\\
&\psi(\{1, 0, 1, 1, \ldots\}) = 1,\\
\psi(\{0, -1, &-\frac{1}{3}, -\frac{1}{4}, \ldots\}) = 0,\\
&\psi \in (\ell_\infty)'_+.
\end{aligned} \tag{1.8}
$$

When the vector space ℓ_∞ is equipped with its norm topology, its positive cone has a non-empty interior The infinite dimensional version of Slater's constraint qualification (Theorem 2.1) applies and there is zero duality gap with the primal program. However, dual functionals feasible to (1.8) may no longer be countably additive. Let $\bar{\psi}$ be a dual functional over ℓ_∞ that satisfies $\bar{\psi}(\{a_i\}_{i=1}^\infty) := \lim_{i\to\infty} a_i$ for every convergent sequence $\{a_i\}_{i=1}^\infty$. Such a dual functional is guaranteed to exist (for details, see Lemmas 16.29 and 16.30 in Aliprantis and Border [2]). Observe also that $\bar{\psi}$ is feasible to (1.8) and its dual objective value, $\bar{\psi}(\{-1, 0, 0, \ldots\}) = 0$, is equal to the optimal value of the primal. Thus, $\bar{\psi}$ is an optimal dual functional and there is zero duality gap. ◁

The optimality of $\bar{\psi}$ in the previous example highlights three serious issues that are not present in finite dimensions. First, $\bar{\psi}$ is difficult to characterize precisely. The only structure we have specified is that it is linear and acts like a "limit evaluator" on elements of ℓ_∞ that converge. Outside of the subspace of convergent sequences, little is known about how $\bar{\psi}$ operates.

Second, $\bar{\psi}$ fails countable additivity. Consider the sequence $(1, 1, \ldots)$ and observe that $1 = \bar{\psi}((1, 1, \ldots)) \neq \sum_{i=1}^\infty 1 \cdot \bar{\psi}(e_i) = 0$. This implies that we lose the familiar interpretation from finite dimensions that there exists a dual price $\bar{\psi}(e_i) = \bar{\psi}_i$ on the ith constraint so that if each constraint i is perturbed by sufficiently small ϵ_i, then the change in objective is $\sum_{i \in \mathbb{N}} \bar{\psi}_i \epsilon_i$.

Third, the dual (1.8) is not analogous to the finite linear programming dual. The constraints are not defined using the "transpose" of the original constraint matrix and instead are expressed in terms of dual functionals that are not necessarily countably additive.

Moreover, Example 1.2 illustrates how the choice of the vector space for the primal constraint space affects the structure and interpretation of the dual. The dual price vectors for (1.7) are easily characterized – they are finite support vectors. However, the interior of the positive cone of the constraint space is empty and there is a duality gap. Alternatively, choosing a constraint space with a positive cone that has a nonempty interior generates the dual program (1.8) with zero duality gap. However, the resulting structure of the dual problem and the structure of the optimal dual functionals are "undesirable."

In the finite case, dual functionals can always be interpreted as a price on each constraint; however Example 1.2 demonstrates this does not generalize to infinite dimensions. If countable additivity holds then this nice interpretation does carry over. In the general setting of Riesz spaces such "undesirable" dual functionals fail to satisfy the key property of *σ-order continuity*, which is equivalent to a generalized notion of countable additivity.

1.2. Our Contributions. Our main results show that the interplay between the existence of interior points and singular dual functionals observed in Example 1.2 is not an accident. Our results apply to constraint spaces that are infinite dimensional Riesz spaces. We make no topological assumptions and work with the algebraic notion of core points (defined in Section 2) rather than interior points. We show that if the positive cone of the constraint space has a core point then there is zero duality gap with the algebraic dual. Moreover, in a broad class of spaces – Riesz spaces that satisfy either σ-order completeness or the projection property – if the positive cone of the constraint space has a core point then singular dual functionals exist. We call this phenomenon the *Slater conundrum*. On the one hand, the existence of a core point ensures a zero duality gap (a desirable property), but on the other hand, existence of a core point implies the existence of singular dual functionals (an undesirable property).

This approach borrows concepts from geometric functional analysis (see for instance, Holmes [20]) and connects them to concepts in Riesz space theory (see for instance, Aliprantis and Border [2]). Proposition 2.5 unites the concept of a *core point* from geometric functional analysis to the concept of an *order unit* in Riesz spaces. This provides a bridge between two streams of literature that, to the authors' knowledge, produces a novel approach to the study of infinite dimensional optimization problems.

Corollary 2.2 provides a constraint qualification that relies on the existence of a core point in the positive cone of the constraint space. We call this an algebraic constraint qualification because the concept of core can be defined in any ordered vector space. The use of core points for constraint qualifications was first introduced by Rockafellar [29]. However, the condition given applied to optimization problems over Banach spaces and their Fenchel duals. Our algebraic constraint qualification is an extension of these conditions to include general ordered vector spaces.

An advantage of the algebraic constraint qualification is that every interior point in a locally convex topological vector space is a core point (see Holmes [20]). Therefore, if the algebraic constraint qualification fails to hold, then Slater's constraint qualification in any locally convex topology also fails. This property allows us to investigate which ordered vector spaces have interior points in their positive cones and determine the structure of dual functionals on these spaces.

A main result is Theorem 3.8, where we construct a singular dual functional from an order unit and a sequence of vectors that order converge to that order unit. This result is used to establish the existence of singular dual functionals in general Riesz spaces that need not possess any further topological or completeness properties required by other known results (including our own Theorem 3.13 and Theorem D.11 (see also Wnuk [35])). Next, we show (Theorem 3.13) that the general class of Riesz spaces with order units that are either σ-order complete or satisfy the projection property always have a singular dual functional in the algebraic dual space.

Riesz spaces are ordered vector spaces equipped with a lattice structure (see Section 2.2). We focus on Riesz spaces and not other classes of vector spaces for four reasons. First, an ordered vector space is necessary for constrained optimization. The common notion of being "constrained" is based on the concept of ordering. For instance, in finite-dimensional linear programming, inequality constraints $Ax \geq b$ and nonnegativity constraints $x \geq 0$ are defined by the standard ordering of finite-dimensional Euclidean space.

Second, the order and lattice assumptions endow an infinite dimensional vector space with just enough structure for familiar properties in $\mathbb{R}^n$ to have meaningful analogues. On a Riesz space we can define absolute value and order convergence. Also, the concept of disjointness (Section 2.2), generalizes the idea that a nonzero vector can have both zero and nonzero "components."

Third, the *order dual* of a Riesz space is the most general dual space that needs to be considered in optimization when the constraint space Y is an ordered vector space. Given vector space Y, the largest dual space is the algebraic dual Y'. However, when Y is a Riesz space, the order dual $Y^\sim$ consisting of dual functionals that are order-bounded is natural since it contains all of the positive linear functionals in Y'. By Proposition 2.4 the optimal value of the Lagrangian dual defined over Y' and the optimal value of the Lagrangian dual defined over $Y^\sim$ are equal. This implies that the order-dual structure of $Y^\sim$ can be used without loss of optimality when Y is a Riesz space.

Fourth, Riesz spaces provide the lattice structure which is needed to precisely classify which dual functionals are easy to characterize and interpret, and those that are not (i.e., those that are not countably additive) through the concept of order continuity. In an ordered vector space, the underlying algebraic structure is insufficient to separate countably additive dual functionals from those that are not countably additive.

An alternative to imposing a lattice structure on an ordered vector space is to endow it with a topology. This topological vector space approach is far more common in the optimization literature than the Riesz space approach. We show in Remark 3.4 that it is necessary to impose a lattice structure in order to distinguish the desirable from the undesirable dual functionals. However, because topological thinking is so pervasive, many researchers add the norm topology in addition to the lattice structure and work with Banach lattices (for example, Aliprantis and Burkinshaw [4], Wnuk [35], and Zaanen [36]). A key contribution of this paper is to show that the additional topological structure is not necessary for establishing fundamental results in optimization. Riesz spaces have the minimum structure necessary to characterize the Slater conundrum and establish the conundrum as a fundamental problem endemic to convex optimization. We adhere to the dictum expressed by Duffin and Karlovitz [14] of "the desirability of omitting topological considerations" from a position of both enhanced clarity and enhanced generality. Indeed, one of our key results, Theorem 3.13, is more general than similar results obtained for Banach lattices. See Remark 3.15 and Theorem D.11 in Appendix D.

The main result of this paper may be interpreted as being "negative" because we show that the Slater condition for zero duality gap implies the existence of singular functionals which are difficult to characterize. However, all is not lost when it comes to certain specially structured problems. In Section 4 we provide two sets of sufficient conditions that "resolve" the Slater condundrum for linear programs by guaranteeing the existence of optimal dual solutions with no singular component whenever the order dual is solvable. Previous studies, such as Ponstein [26] and Shapiro [34] proposed conditions in the special settings of ℓ_∞ and L_∞. Our conditions are stated over general Riesz spaces and generalize these approaches.

1.3. Literature Review. Our work is related to several streams of literature in economics and optimization. We briefly outline them here.

The Slater conundrum has been explored by others in specific contexts. Rockafeller and Wets [29, 30, 31] and Ponstein [26, 27] observe that L_∞ is the only L_p space with an interior point in its positive cone. They further point out that algebraic dual of L_∞ contains dual functionals that are not countably additive. They conclude that the only L_p space where Slater's constraint qualification can be applied has dual functionals that are difficult to characterize and interpret. Our development avoids topological and measure-theoretic arguments (as used in the L_∞ case) and works in greater generality by focusing on primitive algebraic and order properties. This level of abstraction demonstrates that the Slater conundrum is endemic to infinite dimensional optimization at its very foundation.

In the stochastic programming literature, Rockafellar and Wets [31] emphasize the central role played by singular dual functionals in a complete duality theory for convex stochastic programs. Previously, authors ignored such dual functionals and worked only with countably additive dual functionals. This meant they had to accept the possibility of duality gaps. Although researchers begrudgingly accept that general optimality conditions involve singular dual functionals in an essential way, they are considered to be "unmanageable" from a practical perspective and fastidiously avoided; see for example, Rockafellar and Wets [30]. There has also been a substantial amount of research devoted to finding problem structures that do not have optimal singular dual functionals (see Dempster [13] for a summary). In [26], Ponstein makes a careful study of singular dual functionals in general convex optimization problems with constraint spaces in L_∞. He gives conditions that justify ignoring singular dual functionals without loss of optimality, generalizing some well-known conditions in the stochastic programming literature. Ponstein emphasizes the computational intractability of singular dual functionals.

Other authors have also pursued an order-algebraic approach to optimization while eschewing topological concepts. Holmes's classic monograph [20] on geometric functional analysis sets the stage for order-algebraic approaches by dedicating a large initial part of his monograph to an investigation of functional analysis without reference to topology. Similarly, Anderson and Nash [6] ground their duality theory with a study of algebraic duality before introducing topological notions. In this context they establish weak duality and complementary slackness, as well as lay the algebraic foundation for an extension of the well-known simplex method to infinite dimensional linear programs. Following a similar approach, Shapiro [34] considers conditions that imply zero duality gap for the algebraic dual of infinite conic programs and only later introduces topologies.

Our work, in some ways, parallels developments in theoretical economics on pricing and equilibria in infinite dimensional commodity spaces (see for instance, Aliprantis and Brown [3]). These studies feature Riesz spaces arguments that are akin to ours. While the literature on Riesz spaces is quite extensive, with many quality texts ranging from the introductory level to the advanced (see Aliprantis and Border [2], Aliprantis and Burkinshaw [4], Luxemburg and Zaanen [24] and Zaanen [36, 37]), Riesz spaces are rarely mentioned in optimization theory. A key to our approach is to develop novel results for Riesz spaces and apply these results to infinite dimensional optimization.

The Slater conundrum is driven by constraint qualifications for zero duality gap that require the existence of interior points in the positive cone in the constraint space. Such *interior point* constraint qualifications are not the only approaches to establishing zero duality gap results in infinite dimensional optimization. Researchers have long been aware of the limitations of interior point conditions in the L_p spaces. This awareness has motivated several alternate approaches that are worth briefly mentioning here.

First, some researchers have generalized the concept of an interior of a set. A powerful generalization, the quasi-relative interior, was introduced by Borwein and Lewis [8]. Interior points are contained in the quasi-relative interior, as are core points [8]. For example, while the positive cone $\{x \in L_p[0,1] : x(t) \geq 0 \text{ a.a. } t \in [0,1]\}$ in $L_p[0,1]$ for $1 \leq p < \infty$ has no interior in its norm topology, the quasi-relative interior is $\{x \in L_p[0,1] : x(t) > 0 \text{ a.a. } t \in [0,1]\}$ [8]. Unfortunately, the direct extension of Slater's constraint qualification using the expanded set of quasi-relative interior points does not hold. The existence of a feasible point mapping to a quasi-relative interior point in a positive cone is not sufficient to establish zero duality gap without additional assumptions (see Appendix B for a concrete example). For instance, the constraint qualification presented in Borwein and Lewis [8] requires that the constraint space be finite dimensional. A variety of constraint qualifications based on the quasi-relative interior were later introduced (see for example, Boţ [9] and Grad [18]). By considering the structure of the positive cone, rather than the topological structure of a vector space, our approach avoids the need for additional assumptions by focusing attention on vector spaces where core points exist.

Second, many researchers have focused on topological notions such as closeness, boundedness and compactness to drive duality results, rather than interior points [6, 9, 19]. This approach is particularly useful when interior points are known not to exist. The drawback is that these conditions are generally thought to be difficult to verify in practice, despite their theoretical elegance.

Finally, others have established zero duality gap results using limiting arguments and finite approximations in combination with the basic duality results for finite dimensional linear programming. These include papers on separated continuous linear programs (Pullan [28]) and countably infinite linear programs (Ghate and Smith [17] and Romeijn et al. [33]). These studies avoid certain topological arguments by leveraging duality results from finite dimensional linear programming and careful reasoning about limiting behavior. However, our approach is quite different. We consider general inequality-constrained convex programs (not just linear programs) and do not derive our results from duality results in the finite dimensional setting.

The remainder of the paper is structured as follows. In Section 2.1 we give a constraint qualification that is sufficient for a zero duality gap between the primal and the algebraic dual in any ordered vector space. Section 2.2 is a very brief tutorial on Riesz spaces and provides the necessary background material. In Section 2.3 we show that the algebraic dual and order dual (a Riesz space concept) are equivalent on the positive cone and that a core point corresponds to an order unit (a Riesz space concept). The main results of the paper are in Section 3 where we establish the Slater conundrum: that Slater points lead to bad (singular) dual functionals in the most general setting possible. Section 4 describes two sets of sufficient conditions for "working around" the Slater conundrum in specially structured problems.

We provide several appendices to supplement our results. Appendix A provides a proof of the results in Section 2.1. Appendix B discusses the differences between our algebraic constraint qualification and the constraint qualifications based on the quasi-relative interior. Appendix C includes definitions and results in Riesz space theory that are required in the proofs of some of our results. Appendix D examines the Slater conundrum from the viewpoint of Banach lattices.

2. Optimization in Ordered Vector Spaces.

2.1. Lagrangian Duality. We first review some of the notation, definitions, and concepts of convex optimization and Lagrangian duality. Consider an *ordered vector space* Y containing a pointed, convex cone P. The cone P, called the *positive cone*, defines the vector space ordering $\succeq_P$, with $y \succeq_P \bar{y}$ iff $y - \bar{y} \in P$. The notation $y \succ_P \bar{y}$ indicates that $y - \bar{y} \in P$ and $y \neq \bar{y}$. Other authors have used $y \succ_P \theta_Y$ to mean that y lies in the interior of the cone P. We avoid this usage in favor of explictly stating when a vector lies in the interior of a set.

Consider the following inequality constrained convex program,

$$\text{(CP)} \qquad \begin{aligned} \inf \;\; & f(x) \\ \text{s.t.} \;\; & G(x) \preceq_P \theta_Y \\ & x \in \Omega, \end{aligned}$$

where Ω is a convex set contained in a vector space X, $f : \Omega \to \mathbb{R}$ is a convex functional with domain Ω and $G : \Omega \to Y$ is a P-convex map from Ω into the ordered vector space Y and θ_Y represents the zero element of Y.

Let Y' denote *algebraic dual* of Y, set of linear functionals over Y. Take $\psi \in Y'$. The evaluation $\psi(y)$ of ψ at $y \in Y$ is alternatively denoted by $\langle y, \psi \rangle$; that is, $\langle y, \psi \rangle = \psi(y)$. We use the evaluation notation ($\psi(y)$) and the dual pairing notation ($\langle y, \psi \rangle$) interchangeably and favor the notation that lends the greatest clarity to a given expression. The *algebraic dual cone* of P is $P' := \{\psi \in Y' : \langle y, \psi \rangle \geq 0, \; \forall y \in P\}$ and the elements of P' are called *positive dual functionals* on Y. The restriction of the dual cone P' to a subspace W of Y', is denoted by $Q_W = P' \cap W$ and is called the *positive dual cone* with respect to W.

The Lagrangian function, $L : Y' \to \mathbb{R}$ for (CP) is $L(\psi) := \inf_{x \in \Omega} [f(x) + \langle G(x), \psi \rangle]$. Using this definition of $L(\psi)$, a family of dual programs for (CP) is derived as follows. Let W be a subspace of the algebraic dual Y'. The Lagrangian dual program (D_W) of (CP) with respect to W is

$$(D_W) \qquad \begin{aligned} \sup \;\; & L(\psi) \\ \text{s.t.} \;\; & \psi \in Q_W. \end{aligned}$$

The optimal value of an optimization problem $(\cdot)$ is denoted $v(\cdot)$. *Weak duality* holds when the value of the primal program is greater than or equal to the value of the dual. If $v(CP) = v(D_W)$, then the primal and dual programs have *zero duality gap.* As is well-known (see for instance [6]) weak duality always holds for the Lagrangian dual program (D_W) regardless of the choice of W.

Slater's constraint qualification is perhaps the most well-known sufficient condition for zero duality gap between the primal program (CP) and its *topological dual program* (D_{Y^*}). When the constraint space Y is a locally convex topological vector space, its *topological dual* Y^* is the set of dual functionals that are *continuous* in the topology on Y. Slater's constraint qualification states that there is zero duality gap when $-G : \Omega \to Y$ maps a point in Ω to an interior point (in the topology that defines Y^*) of the positive cone P.

Theorem 2.1 (Slater's Constraint Qualification, Ponstein [27], Theorem 3.11.2). *Let Y be a locally convex topological vector space with positive cone P and topological dual Y^*. If there exists an $\bar{x} \in \Omega$ such that $-G(\bar{x}) \in \operatorname{int}(P)$, then there is an optimal dual solution $\bar{\psi} \in Y^*$ and $v(CP) = v(D_{Y^*})$.*

The set of interior points, $\operatorname{int}(P)$, obviously depends upon the selection of the locally convex topology. The proof of Theorem 2.1 uses the existence of an interior point to construct separating hyperplanes from dual funtionals that are continuous in the topology on Y. Therefore, one would like select a locally convex topology on Y such that: 1) *every* dual functional defined on Y is continuous so that the set of dual functionals is large for the goal of closing the duality gap (and thus the topological and algebraic dual are the same, $Y' = Y^*$) and 2) the set $\text{int}(P)$ is the largest possible set of interior points. These two goals are achieved by using a locally convex topology defined by *core points.*

Given a vector space Y and a subset $A \subseteq Y$, a point $a \in A$ is a *core point* of A if for every $y \in Y$, there exists an $\epsilon > 0$ such that $a + \lambda y \in A$ for all $0 \leq \lambda \leq \epsilon$. The set of core points of A is denoted $\text{cor}(A)$.

Corollary 2.2 (Algebraic Constraint Qualification). *Let Y be a vector space with positive cone P and algebraic dual $Y^{'}$. If there exists an $\bar{x} \in \Omega$ such that $-G(\bar{x}) \in \text{cor}(P)$, then there is an optimal dual solution $\bar{\psi} \in Y^{'}$ and $v(CP) = v(D_{Y'})$.*

Corollary 2.2 follows immediately from Theorem 2.1 and the fact that the core points can be used to define is a locally convex topology on Y where $\text{int}(A) = \text{cor}(A)$. For details, see Appendix A. A core point is a purely geometric concept and the beauty of Corollary 2.2 is that it applies to any ordered vector space. This is in keeping with our philosophy to present results in the most general setting possible. Furthermore, all interior points in any locally convex topological vector space are core points; that is, for all subsets $A \subset Y$, $\text{int}(A) \subseteq \text{cor}(A)$ (see Holmes [20]). If P does not have a core point, then P does not have an interior point in any locally convex topology. This implies that the existence of a core point in P is the most general Slater condition possible.

Remark 2.3. Recently, authors have introduced constraint qualifications using quasi-relative interior points and other topological alternatives [8, 9, 18]. Constraint qualifications based on a quasi-relative interior point require additional structure in order to prove that there is zero duality gap. In Appendix B we show that the existence of a quasi-relative interior point is not sufficient to guarantee a zero duality gap with the Lagrangian dual. $\triangleleft$

$$
\begin{tikzcd}
 & \sigma\text{-order complete} \arrow[dr, Rightarrow] & & \\
\text{order complete} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{principal projection property} \arrow[r, Rightarrow] & \text{Archimedean} \\
 & \text{projection property} \arrow[ur, Rightarrow] & &
\end{tikzcd}
$$

Fig. 1. *Main Inclusion Theorem (Theorem 25.1 in Luxemburg and Zaanen [24])*

2.2. Riesz Spaces. We explore the effect that the existence of a core point in the positive cone has on the structure and interpretation of the dual functionals. For reasons outlined in the introduction, Riesz spaces provide a natural setting for this exploration.

We begin with a few basic definitions and concepts from Riesz space theory needed to understand the statements of our results. This includes definitions of the classifications of Riesz spaces in Figure 2.2. This figure gives the relationships among important classes of Riesz spaces. Luxemburg and Zaanen [24] (Theorem 25.1) refer to these relationships as the *main inclusion theorem.* The main inclusion theorem is used to gain a precise understanding of our contributions in Section 3 (See Remark 3.14). Additional results on Riesz spaces used in proofs, but not needed to understand the statements of our results, are found in Appendix C. For those interested in more details, Chapters

8 and 9 of Aliprantis and Border [2] provide a thorough introduction to Riesz spaces.

An ordered vector space E is a Riesz space if the vector space is also a lattice; that is, each pair of vectors $x, y \in E$ has a supremum denoted $x \vee y$ and an infimum denoted $x \wedge y$. Common examples of Riesz spaces are in Example 8.1 in Aliprantis and Border [2]) and include the L_p spaces and spaces of continuous functions. Let E be a Riesz space ordered by the positive cone E^+. Let θ_E denote the zero vector of E. For each vector $x \in E$, the positive part x^+, the negative part x^- and the absolute value $|x|$ are $x^+ := x \vee \theta_E$, $x^- := x \wedge \theta_E$ and $|x| := x \vee (-x)$. Two vectors $x, y \in E$ are *disjoint*, denoted $x \perp y$, whenever $|x| \wedge |y| = \theta_E$. A set of vectors are *pairwise disjoint* if each pair of distinct vectors are disjoint. Given a subset of a Riesz space $S \subseteq E$, its *disjoint complement* is $S^d := \{y \in E : x \perp y \text{ for all } x \in S\}$. A set S is order bounded from above if there exists an upper bound $u \in E$, such that $x \preceq u$ for all $x \in S$. Similarly, a set is order bounded from below if there exists a lower bound and is *order bounded* if the set is order bounded both from above and below. A set S is *solid* when $|y| \preceq |x|$ and $x \in S$ imply that $y \in S$.

A Riesz space E is *order complete* (sometimes called Dedekind complete) if every non-empty subset of E that is order bounded from above has a supremum. Similarly, a Riesz space is *σ-order complete* if every countable subset with an upper bound has a supremum. The L_p spaces for $1 \leq p < \infty$ are σ-order complete.

Let $\{x_\alpha\}$ be a net of vectors in E and let $\{x_n\}$ represent a sequence. The notation $x_\alpha \uparrow\preceq x$ means that $\{x_\alpha\}$ is order bounded from above by x. When $\sup x_\alpha = x$, we write $x_\alpha \uparrow x$. Define $x_\alpha \downarrow x$ similarly. A Riesz space E is *Archimedean* if $\frac{1}{n}x \downarrow \theta$ for each $x \succeq \theta_E$.

A net $\{x_\alpha\}$ in a Riesz space E *converges in order* to x denoted as $x_\alpha \xrightarrow{\circ} x$ if and only if there exists a net $\{y_\alpha\}$ with the same directed set such that $|x_\alpha - x| \preceq y_\alpha$ for each α and $y_\alpha \downarrow \theta_E$. Note that the notion of order convergence involves the absolute value, and thus cannot be defined in an ordered vector space without a lattice structure. A subset S in E is *order closed* if for any net in S with $x_\alpha \xrightarrow{\circ} x \in E$ has $x \in S$.

A vector subspace of a Riesz space E is a *Riesz subspace* if it is closed under the lattice operations of E. A solid Riesz subspace is called an *ideal*. A *principal ideal* $E_x \subseteq E$ is an ideal generated by a vector $x \in E$ and is defined as $E_x := \{y \in E : \exists \lambda > 0 \text{ s.t. } |y| \preceq \lambda |x|\}$. A band is an ideal that is order closed. The band B_x which consists of the order closure of E_x is called the *principal band* generated by $x \in E$. A band B is called a *projection band* if $E = B \oplus B^d$ where $\oplus$ denotes a direct sum of vector subspaces, meaning that every element x of E can be written uniquely as $x = y + z$ with $y \in B$ and $z \in B^d$ and $|y| \wedge |z| = \theta$. A Riesz space E has the *projection property* if every band is a projection band. A Riesz space E has the principal projection property if every principal band is a projection band. Let B be a projection band in a Riesz Space E. By definition of projection band, $E = B \oplus B^d$ and for every $x \in E$ there exists an $x_1 \in B$ and an $x_2 \in B^d$ such that $x = x_1 + x_2$. Let $P_B : E \to B$ be defined as $P_B(x) := x_1$.

A dual functional $\psi : E \to \mathbb{R}$ is *order bounded* if it maps order bounded sets in E to order bounded sets in $\mathbb{R}$. The set of all order bounded dual functionals on a Riesz space E is called the *order dual* of E and is denoted $E^\sim$. A dual functional $\psi \in E^\sim$ on a Riesz space E is *order continuous* if $\psi(x_\alpha) \to 0$ for all nets $\{x_\alpha\}$ that order converge to θ_E. A dual functional $\psi \in E^\sim$ on a Riesz space E is *σ-order continuous* if $\psi(x_n) \to 0$ for all sequences $\{x_n\}$ that order converge to θ_E. The set of dual functionals that are σ-order continuous $E_c^\sim$ form a subspace of the order dual called the *σ-order continuous dual*. The order dual can be expressed as the direct sum of $E_c^\sim$ and its complementary disjoint subspace $E_s^\sim := (E_c^\sim)^d$; that is, $E_c^\sim \oplus E_s^\sim = E^\sim$ (see Theorem 8.28 of Aliprantis and Border [2]). The dual functionals $\psi \in E_s^\sim$ are called *singular dual functionals*.

2.3. An Order-Algebraic Approach. Consider the convex, inequality constrained program (CP) where the constraint space Y is an ordered vector space. When Y is also a Riesz space, then its order dual $Y^{\sim}$ generates the Lagrangian dual program $(D_{Y^{\sim}})$. The added structure of the order dual allows us to further characterize and interpret dual functionals. Corollary 2.2 applies only to the algebraic dual program. In this section, we show this result also applies to the order dual program. The resulting order-algebraic approach connects the theory developed in general ordered vector spaces to the structure and interpretations available in Riesz spaces.

By definition, the order dual of a Riesz space Y contains only order bounded dual functionals and is a subset of the algebraic dual. The feasible region of the algebraic dual program $(D_{Y'})$ consists of positive dual functionals. The following proposition shows that the order dual program and algebraic dual program have the same feasible region and are therefore equivalent.

Proposition 2.4. *For the convex program (CP), the value of the algebraic dual equals the value of the order dual; that is, $v(D_{Y'}) = v(D_{Y^{\sim}})$.*

Proof. It suffices to show that the feasible regions of $(D_{Y'})$ and $(D_{Y^{\sim}})$ are equal; that is, $P' = Q_{Y^{\sim}}$ where $Q_{Y^{\sim}} = P' \cap Y^{\sim}$. Clearly $Q_{Y^{\sim}} \subseteq P'$ so it suffices to show $Q_{Y^{\sim}} \supseteq P'$. Let $\psi \in P'$, that is ψ is a positive dual functional over Y. Let $x, y, z \in Y$ and $x \preceq y \preceq z$. Then, $\psi(y - x) \geq 0$ and $\psi(z - y) \geq 0$. This implies $\psi(x) \leq \psi(y) \leq \psi(z)$ and the dual functional ψ is order bounded. That is, $\psi \in Y^{\sim}$. This implies $\psi \in P' \cap Y^{\sim} = Q_{Y^{\sim}}$. We conclude $P' = Q_{Y^{\sim}}$. □

Proposition 2.4 provides a crucial link between the algebraic constraint qualification and the order dual. Restricting the space of all dual functionals to the order dual does not create a duality gap with the primal. Therefore, if a core point satisfies the algebraic constraint qualification then $v(CP) = v(D_{Y'})$ and Proposition 2.4 implies zero duality gap between the primal and the order dual.

Core points of the positive cone relate to a fundamental concept in the theory of Riesz spaces called order units. An element $e \succ \theta_E$ in a Riesz space E is an *order unit* if for each $x \in E$ there exists a $\lambda > 0$ such that $|x| \leq \lambda e$. The equivalence between order units and core points is given below. This result was also known by Aliprantis and Tourky [5]. However, Aliprantis and Tourky do not use this result within the context of optimality conditions. We provide a proof for the sake of completeness.

Proposition 2.5. *If E is a Riesz space and $e \succ \theta_E$, then e is an order unit of E if and only if e is a core point of the positive cone E^+.*

Proof. Assume that e is an order unit and let z be an an arbitrary element of E. We show there exists an $\epsilon > 0$ such that $e + \lambda z \in E^+$ for all $\lambda \in [0, \epsilon]$. If e is an order unit then by definition there exists an $\alpha > 0$, such that $|z| \preceq \alpha e$. Let $\epsilon = 1/\alpha$. Then $\lambda |z| \preceq e$ for all $\lambda \in [0, \epsilon]$. This implies $-\lambda z \preceq e$ and therefore $\theta_E \preceq e + \lambda z$. Therefore, $e + \lambda z \in E^+$ for all $\lambda \in [0, \epsilon]$ and by definition, e is a core point of E^+. Next, assume e is a core point of E^+ and let z be any element in E. Then there exists a $\epsilon_+ > 0$ such that $e + \lambda z \in E^+$ for all $\lambda_+ \in [0, \epsilon_+]$. If $e + \lambda_+ z \in E^+$, then $e + \lambda_+ z \succeq \theta_E$ and therefore $-z \preceq \frac{1}{\lambda_+} e$. Applying the same logic to $-z$ there exists a λ_- that gives $z \preceq \frac{1}{\lambda_-} e$. Take $\lambda = \min\{\lambda_+, \lambda_-\}$. Then $-z \preceq \frac{1}{\lambda} e$ and $z \preceq \frac{1}{\lambda} e$ implies $-z \vee z \preceq \frac{1}{\lambda} e$. By definition $|z| = z \vee -z$ so $|z| \preceq \frac{1}{\lambda} e$ and e is an order unit of E. □

Proposition 2.4 and Proposition 2.5 provide the foundations for our order-algebraic approach. Since the value of the order dual and the algebraic dual are equal, the algebraic constraint qualification provides a sufficient condition for zero duality gap between the primal program and the order dual. In Riesz spaces, core points of the positive cone are order units. Therefore, only Riesz spaces that contain order units can satisfy Corollary 2.2.

3. The Slater Conundrum. This section contains the main results of the paper. We show for a broad class of Riesz spaces (σ-order complete spaces) that if the positive cone has a core point, then it is necessary to either show that the optimal dual functional does not have a singular component, or somehow characterize the singular components. To our knowledge, this has never been shown in the broad contexts of Theorems 3.8, 3.12, and 3.13.

3.1. Duality and Countable Additivity. In this subsection we expand a theme in Example 1.2 regarding the undesirability of dual functionals that are not countably additive. Countable additivity (defined in Example 1.2 for sequence spaces) extends the familiar interpretations of dual functionals to infinite dimensional Riesz spaces. In fact, the σ-order continuous dual functionals introduced in Section 2.2 are countably additive. We now make precise the concept of countable additivity for arbitrary Riesz spaces.

Given any sequence $\{a_i\}_{i=1}^n$ of real numbers, the limit of partial sums $\sum_{i=1}^n a_i$ is written as $\sum_{i=1}^\infty a_i$ whenever the limit of partial sums exists. Loosely speaking, the reason a lattice structure is added to an ordered vector space is to give the corresponding Riesz space properties that mimic the real numbers as closely as possible. Given a lattice structure, order convergence can be defined, and it is similar to convergence of real numbers. Now assume that the a_i are vectors in an arbitrary Riesz space E. If $x_n \overset{\circ}{\to} \bar{x}$ where x_n is the partial sum $x_n = \sum_{i=1}^n a_i$ we follow the common practice used for real numbers and write $\bar{x} := \sum_{i=1}^\infty a_i$.

Let ψ be a σ-order continuous dual functional defined on E. If $x_n \overset{\circ}{\to} \bar{x}$, then $(x_n - \bar{x}) \overset{\circ}{\to} \theta_E$ and it follows from the definition of σ-order continuity that

$$\psi\left(\sum_{i=1}^\infty a_i\right) = \psi(\bar{x}) = \lim_{n\to\infty} \psi(x_n) = \lim_{n\to\infty} \psi\left(\sum_{i=1}^n a_i\right) = \lim_{n\to\infty} \sum_{i=1}^n \psi(a_i) = \sum_{i=1}^\infty \psi(a_i)$$

where the last equality follows from the fact that the sequence of partial sums of real numbers, $\sum_{i=1}^n \psi(a_i)$, converges to a real number $\psi(\bar{x})$. Since $\psi\left(\sum_{i=1}^\infty a_i\right) = \sum_{i=1}^\infty \psi(a_i)$ we say that ψ is *countably additive.*

Likewise, if a dual functional is countably additive it is σ-order continuous. Assume ψ is a countably additive dual functional on a Riesz space E and $\{x_n\}_{n=1}^\infty$ is a sequence in E such that $x_n \overset{\circ}{\to} \theta_E$. Define a new sequence $\{a_i\}_{i=1}^\infty$ from $\{x_i\}_{i=1}^\infty$ by $a_1 = x_1$ and $a_i = x_i - x_{i-1}$ for $i \geq 2$. Then $x_n = \sum_{i=1}^n a_i$ and $x_n \overset{\circ}{\to} \theta_E$ implies that the sequence of partial sums $\sum_{i=1}^n a_i$ order converges to θ_E. Since ψ is countably additive $\psi\left(\sum_{i=1}^\infty a_i\right) = \sum_{i=1}^\infty \psi(a_i) = \psi(\theta_E)$ and this implies

$$\lim_{n\to\infty} \psi(x_n) = \lim_{n\to\infty} \psi(\sum_{i=1}^n a_i) = \lim_{n\to\infty} \sum_{i=1}^n \psi(a_i) = \sum_{i=1}^\infty \psi(a_i) = \psi(\sum_{i=1}^\infty a_i) = \psi(\theta_E).$$

Then $\lim_{n\to\infty} \psi(x_n) = \psi(\theta_E)$ and ψ is σ-order continuous. Therefore countable additivity and σ-order continuity are equivalent and we have shown:

Proposition 3.1. *If ψ is a dual functional on E, then ψ is σ-order continuous if and only if it is countably additive.*

The next result shows all the dual functionals in the order dual $E^\sim$ are countably additive if and only if the order dual does not contain any nonzero singular dual functionals. Recall from Section 2.2 that a dual functional ψ in the order dual $E^\sim$ of Riesz space E can be written as the sum of a σ-order continuous dual functional $\psi_c \in E_c^\sim$ and a singular dual functional $\psi_s \in E_s^\sim$; that is, $\psi = \psi_c + \psi_s$ where ψ_c and ψ_s are unique.

Theorem 3.2. *Let E be a Riesz space. Then, $E^\sim$ contains a dual functional that is not countably additive if and only if $E^\sim$ contains a nonzero singular dual functional.*

Proof. ($\Leftarrow$) Suppose $E^\sim$ contains a nonzero singular dual functional $\psi \in E^\sim_s$. Since $E^\sim_s$ and $E^\sim_c$ are orthogonal, $E^\sim_c \cap E^\sim_s = \{\theta_{E'}\}$ and so $\psi \notin E^\sim_c$. Then by Proposition 3.1 ψ is not countably additive. ($\Rightarrow$) Suppose $E^\sim$ contains a dual functional ψ that is not countably additive. By Theorem 8.28 in Aliprantis and Border [2], $\psi = \psi_c + \psi_s$ for $\psi_c \in E^\sim_c$ and $\psi_s \in E^\sim_s$. Then $\psi \notin E^\sim_c$ implies $\psi_s \neq \theta_{E'}$ and $E^\sim_s$ contains a nonzero element. □

By Theorem 3.2 singular dual functionals are clearly a problem because their existence implies that there are dual functionals in the order dual that are not countably additive. We show in Section 3.2 a tight connection between Riesz spaces E with order units and the order duals $E^\sim$ that have singular dual functionals.

Countable additivity allows us, in many cases, to write an infinite dimensional dual that is analogous to the finite dimensional dual. Consider the special case of *linear programs*. In linear programs with infinite dimensional constraint spaces, σ-order continuity plays an important role in *expressing* the dual in a familiar way. Consider the linear program

$$\begin{aligned} \inf_{x \in X} \quad & \varphi(x) \\ \text{s.t.} \quad & A(x) \succeq_P b \end{aligned} \tag{3.1}$$

where X is a vector space, $b \in Y$ where Y is an ordered vector space with positive cone P, $A : X \to Y$ is a linear map, and φ is a linear functional on X. The algebraic dual (see for instance [6] for details) of (3.1) is

$$\begin{aligned} \sup_{\psi \in Y'} \quad & \psi(b) \\ \text{s.t.} \quad & A'(\psi) = \varphi \\ & \psi \in P', \end{aligned} \tag{3.2}$$

where $A' : Y' \to X'$ is the *algebraic adjoint* of A, defined by $\langle x, A'(\psi)\rangle = \langle A(x), \psi\rangle$. Giving a concrete expression for the adjoint A' is, in general, difficult. However, when the vector spaces are finite dimensional characterizing A' is easy.

If $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ then (3.1) and (3.2) are easily expressed in terms of matrices. The linear map A is characterized by an m by n matrix (abusing notation in the standard way) $A = (a_{ij})$ where $a_{ij} \in \mathbb{R}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ and given $x \in \mathbb{R}^n$, $A(x) = \sum_{j=1}^n a_{\cdot j} x_j$ where $a_{\cdot j}$ is the jth column of matrix A. The algebraic adjoint map is characterized by the matrix transpose $A^\top$. To see this, recall that ψ is countably additive and characterized by the vector $\psi = (\psi_i)_{i=1}^m$ with $\psi_i \in \mathbb{R}$, where for $y \in \mathbb{R}^m$, $\langle y, \psi\rangle = \sum_{i=1}^m y_i \psi_i$. Then

$$\begin{aligned} \langle A(x), \psi\rangle &= \sum_{i=1}^m A(x)_i \psi_i \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) \psi_i = \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} \psi_i \right) x_j = \langle x, A'(\psi)\rangle. \end{aligned} \tag{3.3}$$

Thus the adjoint operator corresponds to the usual matrix transpose, i.e. $\langle x, A'(\psi)\rangle = A^\top \psi(x)$.

The above analysis depends on two fundamental properties: (i) that the dual functional ψ over $\mathbb{R}^m$ is expressed as a real vector ψ in $\mathbb{R}^m$; and (ii) that it is permissible to swap the finite sums in (3.3).

Now consider $X = \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^{\mathbb{N}}$ as in Example 1.2 where countable additivity is

$$\psi\left(\{a_i\}_{i=1}^{\infty}\right) = \psi\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{\infty} \psi\left(a_i e_i\right) = \sum_{i=1}^{\infty} a_i \psi\left(e_i\right) = \sum_{i=1}^{\infty} a_i \psi_i. \tag{3.4}$$

Combining countable additivity of ψ with the fact X is a n-dimensional vector space gives

$$\begin{aligned} \langle A(x), \psi\rangle &= \sum_{i=1}^{\infty} A(x)_i \psi_i \\ &= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} a_{ij} x_j\right)\psi_i = \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} a_{ij}\psi_i\right) x_j = \langle x, A'(\psi)\rangle \end{aligned} \tag{3.5}$$

and once again, even though A has an infinite number of rows, we can write the adjoint as $A' = A^{\top}$. This phenomenon was demonstrated in Example 1.2 where the dual program (1.7) is characterized in the familiar way based on the "transpose" of the primal constraints. It is the absence of singular functionals in $Y^{\sim}$ that allows one to write a dual with this convenient dual representation. However, in (1.8) the dual program did not have a nice characterization because $\bar{\psi}$ was not countably additive.

When both X and Y are infinite dimensional sequence spaces countable additivity is not sufficient to justify writing the adjoint as the transpose matrix. Additional properties on the linear map A are required to apply Fubini's theorem and change the order of summation, as was done in (3.3). Identifying such additional conditions in the case of countably infinite linear programs, where X and Y are both sequence spaces, has been the focus of several studies including Ponstein [26], Romeijn, Smith, and Bean [33] and Ghate [16].

Even in many function spaces countable additivity provides a structure for dual functionals that is analogous to the finite dimensional case. Consider an arbitrary $\sigma-$finite measure space (T, Σ, μ). The set of all real $\mu-$measurable functions is the Riesz space $M(T, \Sigma, \mu)$. An *ideal* of a Riesz space E is a solid Riesz subspace of E. The following theorem shows that for any ideal of $M(T, \Sigma, \mu)$, a dual functional that is σ-order continuous has an appealing structure.

Theorem 3.3 (Zaanen [36], Theorem 86.3). *For any σ-order continuous dual functional ψ on an ideal F of $M(T, \Sigma, \mu)$, there exists a μ-measurable function p on T such that $\psi(y) = \int_{t\in T} p(t)y(t)d\mu$ holds for all $y \in F$. Furthermore, p is $\mu-$almost everywhere uniquely determined.*

Among the ideals of $M(T, \Sigma, \mu)$ are the associated $L_p(T, \Sigma, \mu)$ spaces for $1 \leq p \leq \infty$. Theorem 3.3 provides a large class of Riesz spaces where dual functionals that are σ-order continuous can be characterized by a measurable function p. This function is analogous to the vector $(\psi_i : i \in \mathbb{N})$ in Example 1.2. It assigns a value $p(t)$ to every element $t \in T$, which is interpreted analogously to a shadow price or marginal value. Furthermore, the integral structure in Theorem 3.3 is a convenient representation of the dual functional ψ and aids in expressing the adjoint provided additional properties allow for Fubini's theorem to apply.

Remark 3.4. One might consider adding a topology to an ordered vector space rather than a lattice. However, by Proposition 3.1 countable additivity and order continuity are equivalent and a lattice structure is necessary to define order continuity and therefore distinguish the countable dual functionals from those that are not. The continuous dual functionals defined by a topology *are insufficient to distinguish countable additivity.* For example, by Theorem D.7, when E is a Banach lattice – a Riesz space with a complete norm that is compatible with the lattice structure – the set of norm-continuous dual functionals is the order dual $E^{\sim}$. However, not every element of $E^{\sim}$ is σ-order continuous as we have seen, for example, in the case of ℓ_{∞}. $\triangleleft$

3.2. Order units and the existence of singular dual functionals. In the analysis that follows, we connect the theory of order units and singular dual functionals in Riesz spaces. Recall that order units (via Proposition 2.5) are fundamental to optimization due to their connection to positive core points and the algebraic constraint qualification (Corollary 2.2).

A simple conjecture is that in an infinite dimensional Riesz space, the existence of an order unit implies the existence of a singular dual functional in the order dual. The following examples demonstrates that this is not the case.

Example 3.5 (Luxemburg and Zaanen [24], Example (v) on page 141 and Zaanen [36], Example 103.5). Consider the Riesz space E of all real functions f on an uncountable set T for which there exists a finite number $f(\infty)$ such that, given any $\epsilon > 0$, we have $|f(t) - f(\infty)| > \epsilon$ for at most finitely many $t \in T$. The function $e(t) = 1$ for all $t \in T$ is an order unit of E. However, Zaanen [36] shows that all dual functionals in the order dual of E are σ-order continuous. $\triangleleft$

This example demonstrates the need for additional structure to guarantee the existence of a singular functional. The next example provides insight into structure leveraged in later proofs.

Example 3.6. The space of bounded sequences ℓ_∞ has the order unit $e = (1, 1, 1, \ldots)$. Let $e_i \in \ell_\infty$ have 1 in component i and zero otherwise. Furthermore, define $x_n := \sum_{i=1}^{n} e_i$. Clearly, $x_n \overset{\circ}{\to} e$. Consider the positive dual functional $\bar{\psi}$ defined in Example 1.2. Then $\bar{\psi}(x_n) = 0$ for all n and $\bar{\psi}(e) = 1$. Thus $\bar{\psi}(e)$ is not σ-order continuous and $\bar{\psi} = \bar{\psi}_c + \bar{\psi}_s$ where $\bar{\psi}_s$ is a nonzero singular dual functional in ℓ_∞. $\triangleleft$

This example guides our approach to constructing singular dual functionals from order units in Theorem 3.8 below. The idea is to find a sequence of vectors $\{x_n\}_{n=1}^{\infty}$ where none of the x_n are order units, but $\{x_n\}_{n=1}^{\infty}$ order converges to an order unit $e \succ \theta_E$. The following result on extending dual functionals is used in the argument.

Theorem 3.7 (Krein-Rutman Theorem, Holmes [20], Theorem 6B). *Let X be an ordered vector space with positive cone P and let M be a subspace of X. Assume that $P \cap M$ contains a core point of P. Then any positive linear dual functional ψ on M admits a positive extension to all of X.*

Theorem 3.8. *Let E be an infinite dimensional Riesz space with order unit $e \succ \theta_E$. If there exists an increasing sequence $\{x_n\}$ of non-order units such that $x_n \overset{\circ}{\to} e$, then there exists a positive, nonzero, singular dual functional on E.*

Proof. Without loss, assume that $x_n \succeq \theta_E$ for all n. Otherwise, replace x_n with the sequence $x_n^+ = x_n \vee \theta_E$. None of the x_n^+ are order units and by Lemma 8.15(ii) in Aliprantis and Border [2], $x_n^+ \overset{\circ}{\to} e$. Let M be a subspace of E defined by $M := \operatorname{span}(\{e\} \cup \{x_n\})$. Then every $y \in M$ is represented by a finite set of scalars $\{\lambda_n\}_{n=0}^{N}$ such that $y = \lambda_0 e + \sum_{n=1}^{N} \lambda_n x_n$. For shorthand, let y_x denote $\sum_{n=1}^{N} \lambda_n x_n$.

Claim 1. *For every $y = \lambda_0 e + \sum_{n=1}^{N} \lambda_n x_n$ in M the value of λ_0 in its representation is uniquely determined.*

Proof of Claim 1: We show that the order unit e cannot be represented as a linear combination of the $\{x_n\}$. Assume the opposite; that is, there exists a finite set of scalers $\{\mu_n\}_{n=1}^{N}$ such that $\sum_{n=1}^{N} \mu_n x_n = e$. Let $\overline{\mu} = \max\{|\mu_1|, |\mu_2|, \ldots, |\mu_N|\}$. Since $\{x_n\}$ is an increasing, nonnegative sequence, $\overline{\mu} N x_N \succeq \sum_{n=1}^{N} \mu_n x_n = e$. Next, consider an arbitrary $z \in E$. Since e is an order unit, there exists an $\alpha > 0$ such that $|z| \preceq \alpha e \preceq \alpha \overline{\mu} N x_N$. By definition, this implies that x_N is an order unit, arriving at a contradiction. Therefore, the value of λ_0 in the decomposition of y is uniquely determined. †

Define the functional $\psi_M : M \to \mathbb{R}$ by $\psi_M(y) = \psi_M(\lambda_0 e + y_x) := \lambda_0$ for all $y \in M$. This functional is well-defined by Claim 1. The following claim establishes the properties on ψ_M needed to apply the Krein-Rutman Theorem.

Claim 2. *The functional ψ_M is a positive linear functional on M.*

Proof of Claim 2: Let $y = \lambda_{0y}e + y_x$ and $z = \lambda_{0z}e + z_x$ be vectors in M and let $\alpha, \beta \in \mathbb{R}$. Then $\psi_M(\alpha y + \beta z) = \psi_M((\alpha\lambda_{0y} + \beta\lambda_{0z})e + \alpha y_x + \beta z_x) = \alpha\lambda_{0y} + \beta\lambda_{0z} = \alpha\psi_M(y) + \beta\psi_M(z)$. Hence, ψ_M is a linear functional on M. Next show that ψ_M is a positive functional. Assume otherwise. Then there exists a $y \in E$ such that $y = \lambda_0 e + \sum_{n=1}^{N} \lambda_n x_n$ where $y \succeq \theta_E$ and $\lambda_0 < 0$. Without loss, scale y such that $\lambda_0 = -1$. Then, $\sum_{n=1}^{N} \lambda_n x_n \succeq e$. As in the proof of Claim 1, let $\overline{\lambda} = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_N|\}$. Then, since $\{x_n\}$ is an increasing, nonnegative sequence, $\overline{\lambda} N x_N \succeq e$. However, this implies that x_N is an order unit and yields a contradiction. †

Since $e \in E^+ \cap M$ is an order unit of E, it is a core point of E^+ by Proposition 2.5. Furthermore, Claim 2 states that ψ_M is a positive linear functional on M. Therefore, by the Krein-Rutman Theorem, ψ_M on M extends to a positive dual functional ψ on E. Notice that ψ is not σ-order continuous, since $x_n \overset{\circ}{\to} e$, $\psi(x_n) \to 0$ since $\psi(x_n) = \psi_M(x_n) = 0$, and $\psi(e) = 1$. Therefore, $\psi = \psi_c + \psi_s$ with $\psi_s \neq \theta_E$, since $E^\sim = E_c^\sim \oplus E_s^\sim$. Since $E^\sim$ is a Riesz space (see Theorem 8.24 in [2]) and ψ is positive, Theorem C.4 implies that ψ_s is a positive singular dual functional on E. □

Theorem 3.8 applies to arbitrary infinite dimensional Riesz spaces. To leverage this result, additional structure is added to a Riesz space to guarantee the existence of an order convergent sequence that satisfies the premise of Theorem 3.8. Below, in Theorem 3.13, we prove that all infinite dimensional Riesz spaces with order units that are either σ-order complete or have the projection property Riesz have sufficient additional structure to meet this criterion. However, as the following example demonstrates, this additional structure is not necessary for the existence of singular dual functionals.

Example 3.9 (Luxemburg and Zaanen [24], example (iv) page 140). The Riesz space $\ell_\infty^0(\mathbb{N})$ is the space of all sequences that are constant except for a finite number of components. For example, the sequence $(2, 0, 3, 1, 1, 1 \ldots)$ is an element of $\ell_\infty^0(\mathbb{N})$. Let $e = (1, 1, \ldots)$ be the vector of all ones. Then the set $\{e_i : i = 1, 2, 3, \ldots\} \cup \{e\}$ forms a Hamel basis of $\ell_\infty^0(\mathbb{N})$ and every dual functional can be characterized by assigning values to each vector in the set. The dual functionals on $\ell_\infty^0(\mathbb{N})$ that are not σ-order continuous are those that assign a value of zero to all e_i and a nonzero value to e. Let ψ be such a dual functional. Then $y_n \overset{\circ}{\to} e$ where $y_n = \sum_{i=1}^{n} e_i$, but $\lim_{n\to\infty} \psi(y_n) = 0 \neq \psi(e)$ and so $\ell_\infty^0(\mathbb{N})$ contains a singular dual functional. However, $\ell_\infty^0(\mathbb{N})$ is not σ-order complete and does not have the projection property. First consider $x_1 = (1, 0, 0, \ldots)$, $x_2 = (1, 1/2, 0, 0, \ldots)$, $x_3 = (1, 1/2, 1/3, 0, 0, \ldots)$, $\ldots$. The sequence $\{x_n\}$ is order bounded from above by $e \in \ell_\infty^0(\mathbb{N})$ but clearly has no supremum in $\ell_\infty^0(\mathbb{N})$ and is therefore not σ-order complete. Next consider the band $B = \{x \in \ell_\infty^0(\mathbb{N}) : x(i) = 0, i \,\text{odd}\}$. Then $x \in B$ implies x has a finite number of nonzero even components. Clearly all of the $y \in B^d$ have the property that the even components of y must be zero and this implies every $y \in B^d$ has a finite number of nonzero odd components. Then $e \notin B \oplus B^d$ so B cannot be a projection band. ◃

The proof of Theorem 3.12 shows how to construct, in any infinite dimensional Riesz space with order unit that is either σ-order complete or has the projection property, an increasing sequence $\{x_n\}$ of non-order units such that $x_n \overset{\circ}{\to} e$. Constructing $\{x_n\}$ requires the following lemma, due to Luxemburg and Zaanen [24], used to generate an initial sequence from which we construct $\{x_n\}$.

Lemma 3.10 (Luxemburg and Zaanen [24], Proposition 26.10). *Every infinite dimensional Archimedean Riesz space E contains an infinite set of pairwise disjoint elements.*

The next example uses ℓ_∞ to motivate the steps of the proof of our key result Theorem 3.12.

Example 3.11. Let E be the space of bounded sequences ℓ_∞ with the order unit $e = (1, 1, 1, \ldots)$. Assume Lemma 3.10 generates a sequence of vectors $\{u_n\}$ where for vector u_n, component $2n-1$ is $(1/2)^n$ and the other components are zero. The first three vectors in $\{u_n\}$ are $u_1 = (\frac{1}{2}, 0, \ldots)$, $u_2 = (0, 0, \frac{1}{4}, 0, \ldots)$, and $u_3 = (0, 0, 0, 0, \frac{1}{8}, 0 \ldots)$. Using u_n, construct the sequence $z_n := \sum_{j=1}^{n} u_j$. Then $z_1 = (\frac{1}{2}, 0, \ldots)$, $z_2 = (\frac{1}{2}, 0, \frac{1}{4}, 0, \ldots)$, $z_3 = (\frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0 \ldots)$, ... has the desirable property that it is increasing and none of the z_n are order units. However, this sequence order converges to $\bar{z} = (\frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \frac{1}{16}, \ldots)$, which is not an order unit. At this point, one might be tempted to simply define $u_0 = e - \bar{z}$ and then redefine $\hat{z}_n := u_0 + z_n$. This gives $\hat{z}_1 = (1, 1, \frac{3}{4}, 1, \frac{7}{8}, \ldots)$, $\hat{z}_2 = (1, 1, 1, 1, \frac{7}{8}, \ldots), \ldots$ and $\hat{z}_n \overset{\circ}{\to} e$. Unfortunately, each z_n is an order unit since $u_0 = (\frac{1}{2}, 1, \frac{3}{4}, 1, \frac{7}{8}, \ldots)$ is an order unit and thus the sequence $\{\hat{z}_n\}$ fails the premise of Theorem 3.8. More care is necessary.

Constructing the appropriate sequence of non-order units requires two additional steps. First, project e onto the band generated by the sequence $\{z_n\}$. This projection is well defined and is equal to $P_B(e) := \sup\{e \wedge nz_n : n = 1, 2, \ldots\} = (1, 0, 1, 0, 1, \ldots)$. Define a new sequence $w_n := e \wedge nz_n$. Each w_n has the same support as z_n and each of its positive components increase to 1 as $n \to \infty$, so $w_1 = (\frac{1}{2}, 0, \ldots)$, $w_2 = (1, 0, \frac{1}{2}, 0, \ldots)$, $w_3 = (1, 0, 1, 0, \frac{3}{8}, 0 \ldots)$, ..., and $w_n \overset{\circ}{\to} P_B(e)$. Second, construct the vector $u_0 = e - P_B(e) = (0, 1, 0, 1, 0, \ldots)$. The w_n for each n and u_0 are not order units and the sequence $\{u_0 + w_n\}$ is an increasing sequence of non-order units such that $u_0 + w_n \overset{\circ}{\to} e$. $\triangleleft$

Theorem 3.12. *Assume E is an infinite dimensional Riesz space that is either σ-order complete or has the projection property. If E contains an order unit $e \succ \theta_E$, then E contains an increasing sequence x_n of non-order units with $x_n \overset{\circ}{\to} e$.*

Proof. By hypothesis E is σ-order complete or has the projection property. Therefore E is Archimedean by the main inclusion theorem (see Theorem 25.1 of Luxemburg and Zaanen [24] and our Figure 2.2). Since E is Archimedean and infinite dimensional, E contains an infinite set $\{u_n\}_{n=1}^{\infty}$ of pairwise disjoint elements by Lemma 3.10. By the definition of disjoint, we can assume that this sequence is positive. Since the u_n are pairwise disjoint none are order units by Lemma C.2.

Define a new sequence of vectors $\{z_n\}$ from the sequence of vectors $\{u_n\}$ by

$$z_n := \sum_{k=1}^{n} u_k. \tag{3.6}$$

Since each element of $\{z_n\}$ is a linear combination of the elements of $\{u_n\}$, Lemma C.3 implies that none of the $\{z_n\}$ are order units. Note that $\theta_E \preceq z_n \uparrow$ since each u_k is a positive vector. The sequence $\{z_n\}$ either converges in order to e or it does not. If $\{z_n\}$ does order converge to e the proof is complete with the desired sequence $\{x_n\} := \{z_n\}$.

Assume $\{z_n\}$ does not order converge to e. Define B to be the band generated by $\{z_n\}$. Since E is either $\sigma-$order complete or has the projection property and $\theta_E \preceq z_n \uparrow$, Theorem 28.3 in Luxemburg and Zaanen [24] implies that B is a projection band and

$$P_B(e) = \sup\{e \wedge nz_m : m, n = 1, 2, \ldots\} = \sup\{e \wedge nz_n : n = 1, 2, \ldots\}. \tag{3.7}$$

Define $w_n := e \wedge nz_n$ for $n \in \mathbb{N}$. Clearly, $\theta_E \preceq \{w_n\} \uparrow$ so (3.7) implies $\{w_n\} \uparrow P_B(e) \preceq e$. By definition, $w_n \preceq nz_n$ for each $n \in \mathbb{N}$. Since the z_n are not order units, nz_n are not order units and $\theta_E \preceq w_n \preceq nz_n$ implies that none of the $\{w_n\}$ are order units. If $e = P_B(e)$, the proof is complete with $\{x_n\} := \{w_n\}$, since $\{w_n\}$ is an increasing sequence of non-order units that order converges to e.

Assume $e \neq P_B(e)$. Since $P_B(e) \preceq e$ it follows that $P_B(e) \prec e$ and $u_0 := e - P_B(e) \succ \theta_E$. Consider the sequence $\{u_0 + w_n\}$. Since $\{w_n\} \uparrow P_B(e)$, it follows from the definition of u_0 that

$\{u_0 + w_n\} \uparrow e$. Because B is a projection band, u_0 is disjoint from the $z_1, z_2, \ldots$, that generate B. Then by Lemma C.3, $u_0 + nz_n$ is not an order unit for all n. Finally, observe that

$$\theta_E \prec u_0 + w_n = u_0 + (e \wedge nz_n) = (u_0 + e) \wedge (u_0 + nz_n) \preceq u_0 + nz_n.$$

Since the $u_0 + nz_n$ are not order units, the $u_0 + w_n$ are not order units. Hence the increasing sequence $\{x_n\} := u_0 + w_n$ of non-order units converges to e. This establishes the result. □

THEOREM 3.13. *All infinite dimensional Riesz spaces with order units that are either σ-order complete or have the projection property have non-zero, positive, singular dual functionals.*

Proof. Let E be an infinite dimensional Riesz space with an order unit $e \succ \theta_E$ that is either σ-order complete or has the projection property. By Theorem 3.12 there exists an increasing sequence $\{x_n\}$ of non-order units such that $x_n \xrightarrow{\circ} e$. Then by Theorem 3.8 there exists a non-zero, positive, singular dual functional on E. □

REMARK 3.14. The results in this section provide insight into which classes of Riesz spaces have the property that the existence of an order unit implies the existence of singular dual functionals. Recall the main inclusion theorem of Luxemburg and Zaanen (Theorem 25.1 in [24]) expressed in Figure 2.2. Theorem 3.13 result shows that infinite dimensional Riesz spaces with order unit that are either σ-order continuous or satisfy the projection property have singular dual functionals. However, the condition is not necessary. Indeed, Luxemberg and Zaanan [24] show that the Riesz space in Example 3.9 satisfies the principal projection property but is neither σ-order continuous nor satisfies the projection property. However, we showed it has a singular dual functional.

The Archimedean property, which is implied by the principal projection property, is insufficient to guarantee the existence of singular dual functionals. Luxemburg and Zaanen [24] show that the Riesz space in Example 3.5 is Archimedean (see Example (v) on page 141), but as we have shown, it has no singular dual functionals. However, Luxemburg and Zaanen [24] also show that the Riesz space in Example 3.5 is not $\sigma-$order complete and does not have the projection property. Therefore Example 3.5 does not contradict Theorem 3.13.

An open question remains regarding the connection between the principal projection property and the existence of singular dual functionals. In particular, one conjecture is that the principal projection property is sufficient for the existence of singular dual functionals in infinite dimensional Riesz spaces with order unit. However, the principal projection property is not necessary. Indeed, $C[0, 1]$ is an infinite dimensional Riesz space with order unit that does not satisfy the principal projection property (see Example (v) on page 140 of Luxemburg and Zaanen [24]) but has singular dual functionals (see page 147 of Zaanen [37]).

REMARK 3.15. Theorem 3.13 uses an order-algebraic approach to show that a large class of infinite dimensional Riesz spaces satisfy Theorem 3.8 and therefore have singular functionals in their order duals. A similar, but less general result, using topological methods is provided in Appendix D. There we show that if E is a σ-order complete Banach lattice with an order unit then $E^{\sim}$ admits singular functionals. However, this result requires the additional structure of being a Banach lattice. This is not required in Theorem 3.13. By Theorem 9.28 of Aliprantis and Border [2], any order complete Riesz space E can be equipped with a norm that defines a Banach lattice. However, unlike Theorem 3.13, this requires order completeness as opposed to σ-order completeness. Moreover, Theorem 3.13 also handles the case where the space satisfies the projection property only and is possibly not even σ-order complete. We also reiterate that Theorem 3.8 applies to all infinite dimensional Riesz spaces, providing a general condition for the existence of singular dual functionals. ◁

4. Resolving the Slater Conundrum. Connecting our algebraic constraint qualification based on core points to the existence of singular dual functionals reveals a significant modeling issue in infinite dimensional programming. We show the tradeoff between the sufficiency of interior point conditions for zero duality gap and the difficulties of working with singular dual functionals. The question thus remains how to "get around" the Slater conundrum and still work in spaces like ℓ_∞ and L_∞, which are otherwise desirable for modeling purposes.

We provide sufficient conditions to bypass the conundrum for general infinite dimensional linear programs over Riesz spaces with structure (3.1). Consider

$$\text{(RLP)} \qquad \begin{aligned} \inf_x \ & \langle x, \varphi \rangle \\ \text{s.t. } & A(x) \succeq_{Y^+} b \\ & x \succeq_{X^+} \theta_X \end{aligned}$$

where $A : X \to Y$ is a linear map and X and Y are both infinite-dimensional Riesz spaces where $\varphi \in X^\sim$, $b \in Y$, Y^+ is a pointed convex convex cone with nonempty core in Y, and X^+ is a pointed convex cone in X. When the underlying vector space is clear, we drop the subscripts on the orderings $\succeq_{X^+}$ and $\succeq_{Y^+}$ for ease of notation. The order dual of (RLP) is

$$\text{(RLP}^\sim\text{)} \qquad \begin{aligned} \sup_\psi \ & \langle b, \psi \rangle \\ \text{s.t. } & A^\sim(\psi) \preceq_{X^+} \varphi \\ & \psi \in (Y^+)^\sim \end{aligned}$$

where $A^\sim : Y^\sim \to X^\sim$ is the *order adjoint* of A defined by $\langle x, A^\sim(\psi)\rangle = \langle A(x), \psi\rangle$ for all $x \in X$ and $\psi \in Y^\sim$, and $(Y^+)^\sim$ is the order dual cone of Y^+.

Remark 4.1. The linear program (RLP) requires that $x \in X^+$, a condition not included of previous formulations in this paper. However, this condition is easily relaxed via a standard argument in linear programming. $\triangleleft$

Our investigation is motivated by the following question. Given an instance of (RLP) where there exists an optimal dual solution to (RLP$^\sim$), does there always exist an optimal dual solution that has no singular component? If we can affirmatively answer this, then we say that we have *resolved* the Slater conundrum. One sufficient condition for resolving the conundrum is:

$$\text{If } \psi^* = \psi^*_c + \psi^*_s \text{ is optimal to (RLP}^\sim\text{) with } \psi^*_c \in Y^\sim_c, \psi^*_s \in Y^\sim_s \text{ then } \psi^*_c \text{ is also optimal.} \tag{4.1}$$

In this section we provide two sets of sufficient conditions that guarantee (4.1) holds and thus resolve the Slater conundrum. The first set of conditions is inspired by Theorem 5.1 of Ponstein [26] for problems in ℓ_∞.

Theorem 4.2. *Consider an instance of* (RLP) *where A and b are such that the following two conditions hold: for all positive singular linear functionals $\psi_s \in (Y^+)^\sim_s$*

$$A^\sim(\psi_s) \succeq \theta \tag{4.2}$$

and

$$\langle b, \psi_s \rangle \le 0. \tag{4.3}$$

Then condition (4.1) *holds.*

Proof. Let $\psi^* = \psi_c^* + \psi_s^*$ be an optimal solution to (RLP$^\sim$). The proof proceeds in two steps. We first show that ψ_c^* is itself a feasible dual solution using (4.2). Second, we show that $\langle b, \psi_c^* \rangle \geq \langle b, \psi^* \rangle$ using (4.3). This establishes that ψ_c^* is also an optimal solution and so (4.1) holds. To establish the first step note that $\varphi \succeq A^\sim(\psi^*) = A^\sim(\psi_c^*) + A^\sim(\psi_s^*) \succeq A^\sim(\psi_c)$ where the first inequality follows from the feasibility of ψ^*, the equality follows from the linearity of the order adjoint and the second inequality follows by (4.2). To establish the second step note that $\langle b, \psi^* \rangle = \langle b, \psi_c^* \rangle + \langle b, \psi_s^* \rangle \leq \langle b, \psi_c^* \rangle$ where the equality follows from linearity and the inequality follows from (4.3). □

Ponstein [26] considers problems where $Y = \ell_\infty$ and uses the notion *singular nonpositive* to condition A and b so that (4.2) and (4.3) hold. A sequence $y = (y_1, y_2, \dots) \in \ell_\infty$ is singularly nonpositive if every accumulation point of y is nonpositive. Lemma 4.2 of [26] shows that if $\psi \in (\ell_\infty^\sim)_s$ then $\psi(y) \leq 0$ for all singularly nonpositive $y \in \ell_\infty$. Ponstein considers a linear program of the form (RLP) where $A : X \to \ell_\infty$ is represented by a doubly infinite matrix (also denoted A) and X is a sequence space that contains the unit vectors e^i for $i = 1, 2, \dots$. Theorem 5.1 of [26] then shows that if $-b$ and the columns of A (themselves sequences) are singularly nonpositive then (4.2) and (4.3) hold, thus establishing (4.1) via our Theorem 4.2.

A sufficient condition to guarantee that the columns of A are singularly nonpositive is to require that that the matrix A has finitely many nonzero entries in each column. This sufficient condition was used by Ghate, Romeijn, and Smith [17, 33, 32] in their work on the duality of countably infinite programs. By using this sufficient condition, they were able to ignore the contributions of singular dual functionals.

What can be done to resolve the conundrum when either (4.2) or (4.3) fail? We provide one further set of sufficient conditions, based on a decomposition of the dual problem, that draws partial inspiration from Shapiro's approach to the duality of conic optimization problems over L_∞ in [34]. The idea is to use the orthogonality of $Y_c^\sim$ and $Y_s^\sim$ and assume a particular structure on A that allows the dual problem to be decomposed into a "continuous" subproblem and a "singular" subproblem.

THEOREM 4.3. *Assume the primal problem* (RLP) *is feasible with $\varphi \in X_c^\sim$ and the order adjoint $A^\sim$ satisfies $A^\sim((Y^+)_n^\sim) \subseteq X_n^\sim$ and $A^\sim((Y^+)_s^\sim) \subseteq X_s^\sim$; that is, the order adjoint maps positive σ-order continuous linear functionals to σ-order continuous linear functionals and positive singular linear functionals to singular linear functionals. Then condition* (4.1) *holds.*

Proof. Let $\psi^* = \psi_c^* + \psi_s^*$ be an optimal solution to (RLP$^\sim$). The proof proceeds by decomposing (RLP$^\sim$) into two subproblems, one involving only ψ_c^* and one involving only ψ_s^*. First, we decompose the constraint $\psi^* = \psi_c^* + \psi_s^* \succeq \theta$ in (RLP$^\sim$). Since $\psi_c^* \perp \psi_s^*$, Theorem C.4 implies $\psi_c^* \succeq \theta$ and $\psi_s^* \succeq \theta$. Of course, $\psi_c^* \succeq \theta$ and $\psi_s^* \succeq \theta$ implies $\psi_c^* + \psi_s^* \succeq \theta$, and so these two conditions are equivalent. Second, we decompose the constraint $A^\sim(\psi^*) \preceq \varphi$. Note that $A^\sim(\psi^*) \preceq \varphi$ holds if and only if $\varphi - A^\sim(\psi_c^*) - A^\sim(\psi_s^*) \succeq \theta$. By assumption $\varphi \in X_c^\sim$ and $A^\sim(\psi_c^*) \in X_c^\sim$ since $\psi_c^* \in (Y^+)_c^\sim$ so $\varphi - A^\sim(\psi_c^*) \in X_c^\sim$. Moreover, by assumption $A^\sim(\psi_s^*) \in X_s^\sim$ since $\psi_s^* \in (Y^+)_s^\sim$. Then by Theorem C.4, $\varphi - A^\sim(\psi_c^*) - A^\sim(\psi_s^*) \succeq \theta$ implies $\varphi - A^\sim(\psi_c^*) \succeq \theta$ and $-A^\sim(\psi_s^*) \succeq \theta$.

The above decompositions imply that ψ^* is an optimal solution to (RLP$^\sim$) if and only if ψ_c^* is an optimal solution to

$$
(\mathrm{RLP}_c^\sim) \qquad \begin{aligned} \sup_{\psi} \quad & \langle b, \psi_c \rangle \\ \text{s.t.} \quad & A^\sim(\psi_c) \preceq \varphi \\ & \psi_c \in (Y^+)_c^\sim \end{aligned}
$$

and ψ_s^* is an optimal solution to

$$(\mathrm{RLP}_s^\sim) \qquad \begin{aligned} \sup_{\psi} \quad & \langle b, \psi_s \rangle \\ \text{s.t.} \quad & A^\sim(\psi_s) \preceq \theta \\ & \psi_s \in (Y^+)_s^\sim \end{aligned}$$

with $v(\mathrm{RLP}^\sim) = v(\mathrm{RLP}_c^\sim) + v(\mathrm{RLP}_s^\sim)$.

The feasible region to $(\mathrm{RLP}_s^\sim)$ is a cone since if ψ_s is feasible then $\lambda\psi_s$ is feasible for all $\lambda \geq 0$. This follows directly from the linearity of the adjoint $A^\sim(\lambda\psi_s) = \lambda A^\sim(\psi_s) \preceq \theta$ for all $\lambda \geq 0$. Therefore, if there exists a feasible solution ψ_s to $(\mathrm{RLP}_s^\sim)$ such that $\langle b, \psi_s \rangle > 0$ then $(\mathrm{RLP}_s^\sim)$ is unbounded. But (RLP) is feasible, so $\infty > v(\mathrm{RLP}) \geq v(\mathrm{RLP}^\sim)$ by weak duality. Thus $\langle b, \psi_s^* \rangle = v(\mathrm{RLP}_s^\sim) = 0$ which implies $\psi_s = \theta$ is also optimal solution to $(\mathrm{RLP}_s^\sim)$. Then $v(\mathrm{RLP}^\sim) = v(\mathrm{RLP}_c^\sim)$ and $\psi_c^* + \theta = \psi_c^*$ is also an optimal solution to $(\mathrm{RLP}^\sim)$. This establishes (4.1). □

Remark 4.4. The approaches in Theorem 4.2 and Theorem 4.3 are different and not implied by each other. Theorem 4.3 puts no condition on the right-hand-side b and so is more general in this regard. Theorem 4.2 does not restrict where $A^\sim$ maps continuous or singular linear functionals, and is thus more general in this direction. We leave for future work the implications of Theorem 4.3 for particular optimization problems in spaces such as ℓ_∞ or L_∞. ◁

Appendix A. The Convex Core Topology.

In this appendix we show Corollary 2.2 follows from Theorem 2.1 under a special topology that captures the inherent algebraic structure of the vector space.

Let Y be a vector space. A subset A of Y is *algebraically open* if $\mathrm{cor}(A) = A$. Let $\mathcal{B}$ denote the set of all algebraically open *convex* sets that contain the origin θ_X. The set $\mathcal{B}$ forms the neighborhood basis of the origin for the topology $\tau(\mathcal{B})$ where

$$U \in \tau(\mathcal{B}) \text{ if and only if } \forall y \in U,\ \exists B \in \mathcal{B} \text{ s.t. } y + B \subseteq U. \tag{A.1}$$

Following Day [11] we call $\tau(\mathcal{B})$ the *convex core topology* of Y. It has also been called the *natural topology* by Klee [22] and generates the locally convex topological vector space $(Y, \tau(\mathcal{B}))$.

Since $(Y, \tau(\mathcal{B}))$ is a locally convex topological vector space we can apply Theorem 2.1 to a problem with $(Y, \tau(\mathcal{B}))$ as a constraint space. Corollary 2.2 is then a direct consequence of the following proposition, whose proof is straightforward and thus omitted.

Proposition A.1. *Let Y be a vector space with positive cone P and $\tau(\mathcal{B})$ the natural topology on Y. Then $\mathrm{int}(P) = \mathrm{cor}(P)$ and the topological dual Y^* under the natural topology is the algebraic dual Y'.*

Appendix B. Comparison with quasi-relative interior.

Borwein and Lewis [8] propose the quasi-relative interior as a useful generalization of the notion of interior for constraint qualifications of convex programs. Let X be a topological vector space with topological dual X^* and let $A \subseteq X$ be a convex set. The quasi-relative interior of a set A is $\mathrm{qri}(A) := \{x \in A : \mathrm{cl}\,(\mathrm{cone}(A - x)) \text{ is a linear subspace of } X\}$.

By Corollary 2.2 (CP) has zero duality gap with its Lagrangian dual if there exists an $x \in X$ such that $-G(x) \in \mathrm{cor}(P)$. Example B.1 below shows that when the condition $-G(x) \in \mathrm{cor}(P)$ is replaced with $-G(x) \in \mathrm{qri}(P)$, then there may be a positive duality gap. Thus Corollary 2.2 is not immediately subsumed by results in the literature on constraint qualifications involving the quasi-relative interior. See Example 21.1 in Boţ [9] for an example where the existence of a quasi-relative interior point is not sufficient to guarantee a zero duality gap with the Fenchel dual.

Example B.1. Consider the semi-infinite linear program

$$\begin{aligned} &\min x_1 \\ &-(1/n)x_1 - (1/n)^2 x_2 + (1/n) \leq 0, \quad n = 1, 2, 3, \ldots \end{aligned} \tag{B.1}$$

with constraint space $\ell_2(\mathbb{N})$. An optimal primal solution is $(x_1^*, x_2^*) = (1, 0)$ for an optimal primal value of 1. A feasible dual solution is a positive dual functional ψ that satisfies

$$\psi\left(\{-1/n\}_{n=1}^{\infty}\right) = 1 \tag{B.2}$$

$$\psi\left(\left\{-(1/n)^2\right\}_{n=1}^{\infty}\right) = 0. \tag{B.3}$$

We claim that the algebraic dual is infeasible. By Theorem 2.4 and Theorem D.7 every positive dual functional ψ corresponds to an element $\{\psi_n\}_{n=1}^{\infty}$ in $\ell_2(\mathbb{N})$ with $\psi_n \geq 0$ for all n. Thus, (B.2) and (B.3) amount to $\sum_{n=1}^{\infty} -(1/n)\psi_n = 1$ and $\sum_{n=1}^{\infty} -(1/n)^2\psi_n = 0$. The above inequalities cannot be satisfied by any positive ψ. The first equality requires at least one of the ψ_n to be strictly positive, which violates the second inequality. We conclude that the algebraic dual of (B.1) is infeasible.

Since the primal is feasible and dual is infeasible, (B.1) has an infinite duality gap. However, there exists an $\bar{x}$ such that $-G(\bar{x}) \in \mathrm{qri}(P)$ where G is the linear map defining the constraints of (B.1) and $P = (\ell_2(\mathbb{N}))^+$. Let $\bar{x} = (2, 0)$. Then $\bar{y} = -G(\bar{x})$ is the sequence $\{1/n\}_{n=1}^{\infty}$. Example 3.11(i) of Borwein and Lewis [8] shows that $\mathrm{qri}(P) = \{y \in \ell_2(\mathbb{N}) : y_n > 0 \text{ for all } n\}$. Hence $\bar{y} \in \mathrm{qri}(P)$. $\triangleleft$

Appendix C. Properties of Riesz Spaces.

This appendix contains results about Riesz spaces used in the paper that are not readily found in other references.

Proposition C.1. *Assume E is Riesz space with positive cone E^+. If $x, y, z \in E^+$, then $x \perp (y + z) \Rightarrow x \perp y$ and $x \perp z$.*

Proof. By definition $x \perp (y + z)$ implies that $|x| \wedge |y + z| = \theta_E$. Then $x, y, z \in E^+$ implies that $\theta_E = |x| \wedge |y+z| = x \wedge (y+z)$. Therefore $\theta_E = x \wedge (y+z) \succeq x \wedge y \succeq \theta_E$ and $\theta_E = x \wedge (y+z) \succeq x \wedge z \succeq \theta_E$, which implies that $x \perp y$ and $x \perp z$. ∎

Lemma C.2. *Let x and y be elements of the Riesz space E. If $y \neq \theta_E$ and $x \wedge y = \theta_E$, then x is not an order unit.*

Proof. Prove the contrapositive and assume x is an order unit. Then there exists a $\lambda > 0$ such that $y \preceq \lambda x$. Then $\lambda x \wedge y = y$ and $y \neq \theta_E$ by hypothesis. By Theorem 8.1(ii) in Zaanen [37], this implies that $x \wedge y \neq \theta_E$, i.e. x and y are not disjoint. ∎

Lemma C.3. *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of nonzero disjoint vectors in Riesz space E. No linear combination of $\{u_n\}_{n=1}^{\infty}$ is an order unit in E.*

Proof. Since the u_n for $n \in \mathbb{N}$ are nonzero disjoint vectors, it follows from Lemma C.2 that the u_n are not order units. Let $\overline{u} = \sum_{n=1}^{N} \alpha_n u_n$ be an arbitrary linear combination of the elements of $\{u_n\}$. Then Theorems 8.1(ii)-(iii) in Zaanen [37] imply that $\overline{u} \perp u_{N+1}$. If $\overline{u}$ is an order unit, then there exists a $\lambda > 0$ such that $u_{N+1} \preceq \lambda \overline{u}$. Thus, $\lambda \overline{u}$ and u_{N+1} are not disjoint since $u_{N+1} \wedge \lambda \overline{u} = u_{N+1} \neq \theta$. Then by Theorem 8.1(ii) in Zaanen [37], $\overline{u}$ and u_{N+1} are not disjoint, yielding a contradiction. Since $\overline{u}$ was an arbitrary linear combination of the the elements of $\{u_n\}$, no linear combination of the elements of the u_n is an order unit. □

THEOREM C.4. *(Cone Decomposition) If E is a Riesz space with $y, z \in E$ such that $y \perp z$, then $y + z \succeq \theta$ if and only if $y \succeq \theta$ and $z \succeq \theta$*

Proof. ($\Leftarrow$) If $y \succeq \theta$ and $z \succeq \theta$ then $y + z \succeq \theta$. ($\Rightarrow$) If $y + z \succeq \theta$, then

$$y + z = |y + z| = |y - z|$$

where the second equality follows from Theorem 8.12 (2) of [2] and the fact that y and z are disjoint. Then by Theorem 8.6 (8) of [2],

$$y \wedge z = \frac{1}{2}(y + z - |y - z|)$$

but $y + z = |y + z| = |y - z|$ gives

$$y \wedge z = \frac{1}{2}(y + z - |y - z|) = \frac{1}{2}(|y - z| - |y - z|) = \theta.$$

If $y \wedge z = \theta$, then $y \succeq (y \wedge z) = \theta$ and $z \succeq (y \wedge z) = \theta$. □

Appendix D. Banach Lattices.

Theorem 3.13 uses an order-algebraic approach to prove that infinite dimensional, σ-order complete Riesz spaces with order units have order duals that contain singular dual functionals. Alternatively, similar, but weaker results can be proved using topological methods. First some definitions.

DEFINITION D.1 (Banach lattice). *Let E be a Riesz space with norm $||\cdot||$. We say $||\cdot||$ is a* lattice norm *if for $x, y \in E$ with $|x| \leq |y|$, then $||x|| \leq ||y||$. A Riesz space equipped with a lattice norm is called a* normed Riesz space. *If the lattice norm is norm-complete (that is, Cauchy sequences converge in norm) then E is a* Banach lattice.

DEFINITION D.2 (AM-space). *A lattice norm on a Riesz space E is an M-*norm *if $x, y \succeq \theta_E$ implies $||x \vee y|| = \max\{||x||, ||y||\}$. A Banach lattice with an M-norm is an* AM-space.

DEFINITION D.3 (Order Continuous Norm). *The lattice norm $\|\cdot\|$ is an order continuous norm if $x_\alpha \downarrow \theta_E$ implies $\|x_\alpha\| \downarrow 0$.*

DEFINITION D.4 (σ-Order Continuous Norm). *The lattice norm $\|\cdot\|$ is a σ- order continuous norm if $x_n \downarrow \theta_E$ implies $\|x_n\| \downarrow 0$.*

THEOREM D.5 (Aliprantis and Border [2], Theorem 9.28). *If E is either a Banach lattice or an order complete Riesz space, then for each $x \in E$ the principal ideal E_x, equipped with the norm $||y||_\infty = \inf\{\lambda > 0 : |y| \preceq \lambda|x|\} = \min\{\lambda \geq 0 : |y| \preceq \lambda x\}$ is an AM-space, with order unit $|x|$.*

If E is an order complete Riesz space with an order unit $e \succ \theta_E$, then the principal ideal E_e is all of E. Therefore, Theorem D.5 implies that every order complete Riesz space with order unit is a Banach lattice, indeed an AM-space when equipped with norm topology $||\cdot||_\infty$.

THEOREM D.6 (Aliprantis and Burkinshaw [4], Corollary 4.4). *All lattice norms that make a Riesz space a Banach lattice are equivalent.*

THEOREM D.7 (Aliprantis and Burkinshaw [4], Corollary 4.5). *If E is a Banach lattice then $E^* = E^\sim$ where E^* is the norm dual and $E^\sim$ is the order dual.*

THEOREM D.8 (Aliprantis and Burkinshaw [4], Theorem 4.51; or Wnuk [35], Theorem 1.1). *For an arbitrary σ-order complete Banach lattice E the following statements are equivalent: (i) E*

does not have order continuous norm and (ii) There exists an order bounded disjoint sequence of E^+ that does not converge in norm to zero.

THEOREM D.9 (Zaanen [36], Theorem 103.9). *A Banach lattice E has an order continuous norm if and only if E is σ-order complete and E has a σ-order continuous norm.*

THEOREM D.10 (Zaanen [36], Theorem 102.8). *A Banach lattice E has a σ-order continuous norm if and only if $E^* = E^\sim_c$.*

The logic for Theorem D.11 given below is inspired by page 48 of Wnuk [35].

THEOREM D.11. *If an infinite dimensional vector space E is either a σ-order complete Banach lattice or an order complete Riesz space and E contains an order unit $e \succ \theta_E$, then there exists a nonzero singular functional in the algebraic dual E', the order dual $E^\sim$ and the norm dual E^*.*

Proof. By Theorem D.5, $(E, ||\cdot||_\infty)$ is an AM-space with order unit where $||\cdot||_\infty$ is defined by

$$\|x\|_\infty := \inf\{\lambda > 0 \,:\, |x| \preceq \lambda e\}. \tag{D.1}$$

By Theorem D.6 all lattice norms that make E a Banach lattice are equivalent, and so without loss, take E to be the Banach lattice $E = (E, \|\cdot\|_\infty)$.

By hypothesis, E is infinite dimensional. Then by Lemma 3.10, E contains an infinite sequence $\{x_n\}_{n=1}^\infty$ of pairwise disjoint elements. By definition of disjointness, we can assume this sequence is positive. By Theorem 8.1(ii) in Zaanen [37] the sequence defined as

$$y_n := \frac{x_n}{\|x_n\|_\infty} \tag{D.2}$$

is still positive pairwise disjoint. By the definition of y_n in (D.2), $\|y_n\|_\infty = 1$ which implies by definition of the $\|\cdot\|_\infty$ in (D.1) that $|y_n| \preceq e$. Thus the sequence $\{y_n\}_{n=1}^\infty$ is order bounded by e. Hence $\{y_n\}_{n=1}^\infty$ is an order bounded sequence of positive pairwise disjoint elements that norm converges to 1. Therefore condition (ii) of Theorem D.8 holds and this implies E does not have an order continuous norm.

Since E is σ-order complete without an order continuous norm, Theorem D.9 implies that E does not have a σ-continuous norm. Therefore, Theorem D.10 implies $E^* \neq E^\sim_c$ and there exists a nonzero singular dual functional in the norm dual of E. Since $E^* = E^\sim \subseteq E'$, the norm dual, the order dual and the algebraic dual of E contain a nonzero singular dual functional. □

REFERENCES

[1] D. ADELMAN AND D. KLABJAN, *Duality and existence of optimal policies in generalized joint replenishment*, Mathematics of Operations Research, 30 (2005), pp. 28–50.

[2] C.D. ALIPRANTIS AND K.C. BORDER, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer Verlag, second ed., 2006.

[3] C.D. ALIPRANTIS AND D.J. BROWN, *Equilibria in markets with a Riesz space of commodities*, Journal of Mathematical Economics, 11 (1983), pp. 189–207.

[4] C.D. ALIPRANTIS AND O. BURKINSHAW, *Positive Operators*, vol. 119, Springer, 2006.

[5] C.D. ALIPRANTIS AND R TOURKY, *Cones and Duality*, American Mathematical Society, second ed., 2007.

[6] E.J. ANDERSON AND P. NASH, *Linear Programming in Infinite-Dimensional Spaces: Theory and Applications*, Wiley, 1987.

[7] AMITABH BASU, KIPP MARTIN, AND CHRISTOPHER THOMAS RYAN, *On the sufficiency of finite support duals in semi-infinite linear programming*, Operations Research Letters, 42 (2014), pp. 16–20.

[8] J.M. BORWEIN AND A.S. LEWIS, *Partially finite convex programming, Part I: Quasi relative interiors and duality theory*, Mathematical Programming, 57 (1992), pp. 15–48.

[9] R.I BOŢ, *Conjugate Duality in Convex Optimization*, vol. 637, Springer, 2010.

[10] J.G. CARLSSON, *Dividing a territory among several vehicles*, INFORMS Journal on Computing, (2011).

[11] M.M. Day, *Normed linear spaces*, Springer, 1973.
[12] G. Debreu, *Valuation equilibrium and Pareto optimum*, Proceedings of the National Academy of Sciences of the United States of America, 40 (1954), p. 588.
[13] M.A.H. Dempster, *On stochastic programming II: Dynamic problems under risk?*, Stochastics: An International Journal of Probability and Stochastic Processes, 25 (1988), pp. 15–42.
[14] R.J. Duffin and L.A. Karlovitz, *An infinite linear program with a duality gap*, Management Science, 12 (1965), pp. 122–134.
[15] G. Gallego and G. Van Ryzin, *Optimal dynamic pricing of inventories with stochastic demand over finite horizons*, Management science, 40 (1994), pp. 999–1020.
[16] A. Ghate, *Markov Chains, Game Theory, and Infinite Programming: Three Paradigms for Optimization of Complex Systems*, University of Michigan (thesis), 2006.
[17] A. Ghate and R.L. Smith, *A linear programming approach to non stationary infinite-horizon Markov decision processes*, Operations Research, 61 (2013), pp. 413–425.
[18] A. Grad, *Quasi-relative interior-type constraint qualifications ensuring strong Lagrange duality for optimization problems with cone and affine constraints*, Journal of Mathematical Analysis and Applications, 361 (2010), pp. 86–95.
[19] R.C. Grinold, *Continuous Programming*, PhD thesis, University of California, Berkeley, 1968.
[20] R.B. Holmes, *Geometric Functional Analysis and its Applications*, Springer-Verlag New York, 1975.
[21] D.F. Karney, *Duality gaps in semi-infinite linear programming – an approximation problem*, Mathematical Programming, 20 (1981), pp. 129–143.
[22] V. Klee, *The Euler characteristic in combinatorial geometry*, American Mathematical Monthly, 70 (1963), pp. 119–127.
[23] D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley-Interscience, 1969.
[24] W.A.J. Luxemburg and A.C. Zaanen, *Riesz Spaces I*, North-Holland, 1971.
[25] A.M. Manelli and D.R. Vincent, *Optimal procurement mechanisms*, Econometrica: Journal of the Econometric Society, (1995), pp. 591–620.
[26] J.P. Ponstein, *On the use of purely finitely additive multipliers in mathematical programming*, Journal of Optimization Theory and Applications, 33 (1981), pp. 37–55.
[27] ———, *Approaches to the Theory of Optimization*, vol. 77, Cambridge University Press, 2004.
[28] M.C. Pullan, *An algorithm for a class of continuous linear programs*, SIAM Journal on Control and Optimization, 31 (1993), p. 1558.
[29] R.T. Rockafellar, *Conjugate Duality and Optimization*, SIAM, Philadelphia, PA, 1974.
[30] R.T. Rockafellar and R.J.B. Wets, *Stochastic convex programming: relatively complete recourse and induced feasibility*, SIAM Journal on Control and Optimization, 14 (1976), pp. 574–589.
[31] ———, *Stochastic convex programming: singular multipliers and extended duality, singular multipliers and duality*, Pacific J. Math, 62 (1976), pp. 507–522.
[32] H.E. Romeijn and R.L. Smith, *Shadow prices in infinite-dimensional linear programming*, Mathematics of operations research, 23 (1998), pp. 239–256.
[33] H. E. Romeijn, R.L. Smith, and J.C. Bean, *Duality in infinite dimensional linear programming*, Mathematical programming, 53 (1992), pp. 79–97.
[34] A. Shapiro, *On the duality theory of conic linear programs*, in Semi-Infinite Programming: Recent Advances, M. Goberna and M. Lopez, eds., Springer, 2005, pp. 135–165.
[35] W. Wnuk, *Banach Lattices with Order Continuous Norms*, Polish Scientific Publishers 1999, 1999.
[36] A.C. Zaanen, *Riesz Spaces II*, North-Holland, 1983.
[37] ———, *Introduction To Operator Theory in Riesz Spaces*, Springer, 1997.