Selling bonus actions in video games

BLINDED FOR REVIEW

In the mobile video games industry, a common in-app purchase is for additional "moves" or "time" in single-player puzzle games. We call these in-app purchases *bonus actions*. In some games, bonus actions can only be purchased in advance of attempting a level of the game (pure advance sales (PAS)), yet in other games, bonus actions can only be purchased in a "spot" market that appears when an initial attempt to pass the level fails (pure spot sales (PSS)). Some games offer both advance and spot purchases (hybrid advance sales (HAS)). This paper studies these selling strategies for bonus actions in video games. Such a question is novel to in-app tools selling in video games that cannot be answered by previous advance selling studies focusing on end goods.

We model the selling of bonus actions as a stochastic extensive form game. We show how the distribution of skill among players (i.e., their inherent ability to pass the level), and the inherent randomness of the game, influence selling strategies. For casual games, where low-skill players have a sufficiently high probability of success in each attempt, if the proportion of high-skill players is either sufficiently large or sufficiently small, firms should adopt PAS and shut down the "spot" market. Furthermore, the player welfare maximizing selling strategy is to sell only in the spot market. Hence, no "win-win" strategy exists for casual games. However, PAS can be a win-win for hardcore games, where low-skill players have a sufficiently low success probability for each attempt.

Key words: advanced selling, pricing, video games

History: This article was first submitted in April 2021, returned for revisions in August 2021, and resubmitted in August 2022.

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1. Introduction

Video games are both the largest and fastest-growing segment of the entertainment industry.[1] Mobile games are the largest segment within video games,[2] also representing around 3/4 of total app store revenue on mobile devices in 2018.[3] In 2017, roughly 43 percent of mobile game revenue came from in-app purchases of virtual items and premium content that enhance the in-game experience.[4] Our main interest is level-based single-player puzzle games where in-app purchases of *bonus actions* (for instance, additional moves in a move-limited puzzle game or additional time in a time-based game) are sold to help players finish challenging levels. The qualification *single-player game* means that players are not interacting directly with each other as play in the game proceeds. Examples

[1] https://www.reuters.com/sponsored/article/popularity-of-gaming

[2] https://www.newzoo.com/globalgamesreport

[3] https://www.businessofapps.com/data/app-revenues/

[4] https://www.statista.com/statistics/273120/share-of-worldwide-mobile-app-revenues-by-channel/

include games that are popular in North America like *Candy Crush Saga*, *Cut the Rope*, and *Wordscapes*, as well as games in China like *Happy XiaoXiao Le*. In 2019, more than half of all smartphone users played some type of single-player puzzle game.[5]

Progression in puzzle games can involve a variety of skills—logic, knowledge of language and trivia, hand-eye coordination, quick reaction times, and spatial reasoning—as well as luck. Players are motivated to progress through the puzzles out of a sense of personal accomplishment, competing with other players (for example, advancing through puzzles faster than your friends), or unlocking rewards and additional content.

To provide a specific example, consider a move-limited single-player puzzle game, such as the popular *Candy Crush Saga*. Suppose a player has run out of her initial allotment of (say) 30 moves in attempting a given level.[6] When her last action is expended, the game presents her with an option to purchase five extra moves for $0.99 that she can use to (hopefully) pass the level. Game mechanics stipulate that the five extra moves can only be used in completing the current puzzle and do not carry over if the current puzzle is completed using less than five moves. In other words, each extra move can be used at most once and only in the current puzzle. This "Five Extra Moves" in-app purchase is among the most popular and revenue-generating of *Candy Crush Saga*'s various in-app purchase options.[7]

Players of mobile games typically do not pay for each attempt at passing the puzzle. Returning to the example of *Candy Crush*, the current puzzle could be solved without the need for the five extra moves on a later attempt, costing only time and possibly frustration on the part of the player. Moreover, bonus actions can sometimes be purchased *before* the player attempts the puzzle. For example, the mobile puzzle game *Happy XiaoXiao Le* published by Tencent offers extra moves before an attempt (at an equivalent of $0.10 USD per extra move) *and* after the player has used all of her available free moves (at an equivalent of $0.30 USD per extra move).[8] By contrast, *Candy*

[5] `https://gamingshift.com/most-popular-mobile-game-genres/`

[6] We use the female pronouns "she/her/hers" when referring to players because the majority of mobile puzzle game players are female, see, e.g., `https://quanticfoundry.com/2017/01/19/female-gamers-by-genre/`.

[7] This can be seen at *Candy Crush Saga*'s page in the Apple App Store, which ranks in-app purchases by how much revenue they generate. See `https://apps.apple.com/us/app/candy-crush-saga/id553834731` for the US store. When accessed on 25 July 2022, "Extra Moves" was the top-selling in-app purchase.

[8] These numbers are based on accessing the game in September 2019 and converting prices from *Happy XiaoXiao Le*'s in-game currency to Chinese yuan to US dollars using available exchange rates at that time. These values are, therefore, approximate and vary with time. In particular, the exact value is complicated by several factors, including a varying exchange rate between in-game currency and Chinese yuan due to promotions to purchase in-game currency at a reduced rate, and the possibility of earning in-game currency through playing the game rather than using real currency. There is also the possibility that bonus actions are sold in the spot market at a discount from the "regular" price. Also, prices are complicated by the fact that moves are sold in bundles. The pre-attempt moves are sold in a batch of three while the post-failure moves are sold in a batch of five. In our analysis, we ignore this level of granularity in the pricing decision. Opening up this can of worms would be an interesting direction for future research.

Crush currently does not offer the purchase of "extra moves" until after the player used all of the available "free moves".[9]

Consider again the *Happy XiaoXiao Le* example that offers both "early" and "late" purchases of extra moves in a level that offers 30 moves for free. When considering whether to buy these extra moves in advance, the player weighs buying an extra move at \$0.10 USD, which could potentially be wasted if she finishes the puzzle in, say, 28 moves, versus the risk of having to spend an additional \$0.30 USD per move later if all 30 free moves are expended before passing the level. This "weighing" depends on a combination of the player's skill, utility for passing the level, and the inherent randomness of the level itself.

In other games, certain bonus actions are *only* sold in advance of attempting a level. One example is the "freeze" bonus action in *Scramble with Friends* (a mobile game adaptation of the classical board game *Boggle*) that "freezes" time for 20 seconds at the end of a two-minute attempt. This frozen time cannot be purchased at the end of the original two-minute allotment.

This variety of strategies observed in practice raises interesting questions. In this paper, we take the perspective of the firm that is monetizing the players' efforts to pass levels through the sale of bonus actions. We ask the following:

(Q1) When to sell bonus actions? Two timings are considered: before attempting the game (advance selling) and after attempting the game (spot selling).

(Q2) When to shut down the spot market and only sell bonus actions in advance?

Answers to these questions should depend on the players' characteristics and the nature of the levels themselves. Passing a level is a combination of both skill and luck, and so it is natural to examine how the answer to (Q1) and (Q2) depends on the following two factors:

(F1) the distribution of skill among players, and

(F2) the inherent randomness (or 'entropy') of the level

Regarding (F1), some players have fast reflexes and quick thinking, while others are more methodical or act less instinctively. Our model abstractly considers only two types of players: high-skill players and low-skill players. The bucket of high-skill players play the game regularly and commit themselves to learning the necessary skills for success. A typical high-skill player is a teenage girl competing with her friends to progress quickly through a game. She gives the game concentrated attention, and she uses what could be considerable skills to tackle the puzzles. By contrast, a low-skill player is not so committed to excelling in the game but uses the game to pass the time or ease her mind. An example low-skill player is a mother playing a puzzle game while waiting in line at her child's doctor appointment. She is not bringing her entire mind to the game, her attention

[9] We discuss this strategic design choice in more detail in Section 8.

is split with other activities. Operationally, we model factor (F1) as the ratio of high-skill and low-skill players and the skill difference between high-skill and low-skill players.

Factor (F2) concerns the nature of the level itself. A level may have more or less "randomness" built into its design through the use of random number generators or procedurally generated content. For instance, puzzle games can involve mechanics like cards or dice being randomly drawn or having certain items or play pieces randomly "drop" into play or unpredictably "react" upon manipulation. A low-skill player with a lucky "draw" can sometimes finish a puzzle, whereas even the most skilled of players, if unlucky, can fail. Operationally, we model factor (F2) by parameters that affect the success probabilities for attempts of both high-skill and low-skill players. We formalize a stochastic extensive form game model to study question (Q1) and (Q2) in light of (F1) and (F2).

Positioning of the paper

Although a vast body of literature studies the timing of selling products and services (e.g. Xie and Shugan 2001, Bhargava and Chen 2012), the video-game setting that interests us in this paper does not fit any known settings in the literature. Indeed, the extant literature models the selling of goods that are "ends in themselves" while the bonus action context is about selling goods (bonus actions) that are "means to an end". For short, we refer to goods that are "ends in themselves" as *end goods* and goods that are a "means to an end" as *tools*. Bonus actions are only really useful as a tool to finish a level; their intrinsic value is small. The value of bonus actions depends on the state of the level when the player fails an attempt. The major source of customer utility is the satisfaction of passing the level, not the use of the tool itself. This is a crucial difference.

There are two sources of uncertainty for tools. The first uncertainty is *whether the tool is needed*. The second uncertainty is *how valuable the tool will be at its time of use*. This leads to a fundamentally different extensive-form game from those studied in the extant literature. First, there is only one layer of uncertainty realization for end goods. By contrast, there are *three* layers of uncertainty for tools. These layers correspond precisely to the scenario of using a tool. First, there is uncertainty about whether the tool is needed. Second, there is uncertainty about how hard the job is to complete, even with the tool in hand. Third, there is a chance of success or failure when using the tool. These three levels of uncertainty are entirely natural in the tool setting.

We want to emphasize another conceptual difference between tools and end goods. In the case of an end good, the "favorable state" is associated with an auspicious condition to consume the good. For a tool, the situation is more complex. First, it would be preferable if the player did not need the tool at all. However, this is not a favorable outcome in terms of the *value of the tool*. If a player passes the level without using bonus actions, the bonus actions have proven worthless. From this perspective, a "favorable state", *with respect to* the value of the tool, is when the player *fails* the initial attempt of the level. This is a reversal of the notion of "favorable" as discussed in Xie and Shugan (2001), Bhargava and Chen (2012). Now, *given* the initial attempt at the level leads to failure, the "favorable states" are associated with the ending status of a game, which shows how hard it is to complete, even with the tool in hand. Better ending status is associated with a higher chance that bonus actions lead to passing the level, yielding a greater return for the player. This two-fold, and somewhat contradictory, notion of a "favorable state" is another reason that the vast literature focusing on end goods does not yield appropriate models for the tool setting. Our investigation fills this gap in the literature.

Summary of key findings

We now summarize our key findings. The firm's revenue optimal selling strategy depends on the type of game. In particular, in *hardcore* games where low-skill players have a sufficiently low success probability for each attempt, the firm should always commit to a *pure advance sales (PAS) strategy* where the spot market is shut down, and bonus actions are only sold in advance of level attempts. Removing the spot market allows the firm to charge a higher price in the advance sales market to more players, thus benefiting the firm.[10] In a hardcore game, the spot market will be crowded by low-skill players because it is difficult for these players to pass the level. However, these low-skill players do not value the bonus actions very highly, because they cannot easily pass the level even with additional help in a hardcore game. Hence, the spot market does not generate much revenue for the firm in a hardcore game. Furthermore, the existence of the spot market provides players waiting incentives. Some players will not buy in advance and will wait to see if they get lucky in their initial attempt, leaving themselves in a position in the puzzle where it is worth buying the bonus actions in the spot market. It can, therefore, be more profitable to commit to shutting off the spot market.

On the other hand, in *casual games*, where low-skill players have a sufficiently high probability of success in each attempt, we show that the firm should shut down the spot market and adopt *PAS* if and only if the proportion of high-skill players is either sufficiently large or sufficiently small. Otherwise, the firm should adopt a *hybrid advance selling (HAS) strategy*, where the firm keeps the spot market open and have positive sales in both advance and spot markets. At a high level, this result balances two competing forces. On the one hand, there is the power of having two markets and the ability to price discriminate between high-skill and low-skill players between these two markets. On the other hand, with PAS, there is the value of the firm committing to shutting

[10] Players can observe the timing pattern of bonus actions offered for purchase in a game. So we assume that the firm's commitment is credible and verifiable.

down the spot market, which can motivate players to purchase early by removing their incentive to wait. Intuitively, only when there is a sufficient balance of high-skill players and low-skill players does the benefit from price discrimination dominate.

When there are a large proportion of low-skill players, a high PAS price that attracts only low-skill players can be optimal. An illustrative example here is something like a crossword puzzle game, where skilled players may have little need for bonus actions (and even enjoy the challenge of answering questions without assistance), while low-skill players might be willing to pay a premium to pass difficult puzzles in order to signal intelligence to their friends. By contrast, another strategy is where bonus actions are priced to attract purchases from many of the players. Low-skill players buy bonus actions to increase their chances while high-skill players buy bonus actions to insure against "unlucky" or uncharacteristic mistakes. If priced right, both types of players find it advantageous to purchase early. These examples illustrate the critical importance of factor (F1) in determining the pricing strategy.

Regarding (F2), we show that casual games with a high degree of entropy are more likely to favor PAS strategies. Games of chance (games with high entropy) leave players with a lot of uncertainty as to where they end up after their initial attempt. Since this uncertainty is resolved when the spot market is reached, it can be difficult for firms to capture value in both the advance and the spot markets in the HAS strategy through differential pricing. In PAS, the spot market is eliminated, and so high levels of entropy must be "insured" against *ex-ante*. This yields the managerial implication that game companies should exclusively offer advance purchases in games with a sufficiently high level of randomness, and if they are committed to offering both advance and spot purchases, might earn additional revenue by reducing the overall randomness in their design.

The second dimension of (F1) is the overall range of the skill levels; that is, the degree to which high-skill players are more skilled than low-skill players. We show that as the difference in skill increases, the HAS strategy becomes more attractive for a casual game. A wider range of skills allows for greater opportunities for price discrimination across two markets. The implications of this result are instructive. It is commonly observed that the range of skills for a game changes over time. One possible direction is that skill differences widen over time, as high-skill players find deeper insights into the game that give them a further advantage over low-skill players. Another possible direction is that skill differences narrow over time. This is possible when intuition and raw ability become less important over time as low-skill players gain access to simple, yet effective, strategies. Optimal pricing strategies for bonus actions in a casual game should monitor the overall trend in skill difference and move from PAS to HAS (or vice versa) accordingly.

Lastly, we look at how the practice of selling bonus actions impacts social welfare in the mobile games market. We show that there exists no "win-win" strategy in casual games. That is, there is no selling strategy that results in the highest profit for the firm *and* the highest welfare for players simultaneously. *Pure spot selling (PSS) strategies* maximize player welfare while they are never revenue-optimal for the firm. This raises the potential for policy concerns about this selling practice in the casual games market. Interestingly, selling bonus actions in the spot market is not uncommon in puzzle games (this is the strategy followed by *Candy Crush*) suggesting the possibility that games may follow a strategy a policy of maximizing player welfare with bonus actions to bolster growth and player retention.

Organization of the paper

The paper is organized as follows. The next section contains a literature review, pointing to related literature on intertemporal price discrimination in the context of advance selling, insurance, and warranty design. In this section, we illustrate the novelty of our research questions and results, since existing work does not seriously tackle the question of shutting down the spot market. We also describe the nascent but growing literature on video games. Section 3 presents the basics of our model setup while Sections 4 and 5 describe the decision problems of the players and firm, respectively. Section 6 and Section 7 study the optimal selling strategies for casual and hardcore games. Section 8 explores how the optimal strategy changes as level entropy and skill differences change, and examines how player welfare is affected by the firm's selling strategy. Section 9 concludes. Proofs of all technical results can be found in the e-companion.

2. Literature review

To our knowledge, pricing bonus actions is a novel topic of investigation in the information systems, operations management, and marketing literature. However, there are strong antecedents for analyzing this type of problem, as we now discuss.

The question of whether to sell products in both advance and spot markets has been studied at length in a variety of other settings. Largely speaking, they fall into the general category of *intertemporal price discrimination*, where a seller takes advantage of changing customer preferences over time to increase profits. The classical studies in intertemporal price discrimination like (Stokey 1979) and (Landsberger and Meilijson 1985) focus on a setting where the value consumers have for a product wanes with time. The standard examples here are technology products, where the novelty and operability of the product become less attractive to consumers over time. The key question here is how to price to meet such changing preferences and when to discontinue sales of an aging product. These considerations are not especially salient in the case of bonus actions. A key reason is that the purchase of bonus actions can be separated in time from the consumption of the product. In particular, bonus actions have a specific time window for use that cannot be moved up or delayed. While models for intertemporal price discrimination typically study durable goods, bonus actions are highly perishable and context-specific. Bonus actions can *only* be used at the moment of failure in a given level, no sooner and no later.

Of course, we are not the first to study the scenario where the purchase and consumption of a good or service are separated in time. This is a context well studied in a variety of settings including advance selling of goods (including papers like Dana (1998), Xie and Shugan (2001), Courty (2003), Ma et al. (2019), Wei and Zhang (2018), Cachon and Feldman (2017), Noparumpa et al. (2015), Li and Zhang (2013), Nasiry and Popescu (2012), Shugan and Xie (2000, 2004, 2005), Yu et al. (2015a,b)), insurance markets (including papers like Miller (1972), Loubergé (2013)), warranties on durable goods (including papers like Glickman and Berger (1976), Durugbo (2020)), etc.[11]

In advance selling, the prototypical example is a consumer looking into buying a vacation package some months in advance of the travel date. The consumer's hesitation for buying early is whether they will be in a position or mood to travel once the travel date arrives. While the problem of selling bonus actions shares a related flavor (we sell bonus actions ahead of the potential use), there are several salient differences. We have already discussed the key difference in the introduction: we study the advanced selling of tools, whereas existing papers study the advanced selling of end goods. Further analytical and conceptual comparisons with the two closest papers in the literature to ours (Xie and Shugan (2001), Bhargava and Chen (2012)) are discussed throughout the paper. See, for example, Remarks 2 and 3.[12]

The fact that we only use bonus actions when we "fail" draws similarities with insurance and warranty markets, where the value of insurance (purchased in advance) is only realized when something "bad" happens (in the spot). Moreover, in insurance, the "cost" of the bad outcome is unlikely to be homogeneous in the likelihood of reaching that bad outcome (as we see in the advance selling literature). Those who are prone to injury (in the case of medical insurance) are also likely prone to more *expensive* injuries. There has been consistent interest in insurance in the management sciences over the past decades (see, for instance, Kao et al. (2022), Zhang et al. (2021), Jin et al. (2022) as recent examples and the references therein).

There are important differences between the market for bonus actions and the market for insurance. The most significant difference is probably the fact that in insurance markets, it is not possible

[11] There are related settings like the selling of options or futures in finance, but these roughly follow the same logic as the other examples, so we do not examine them further here.

[12] One difference that may appear to be salient is the fact that bonus actions are digital goods while most applications of intertemporal price discrimination deal with physical goods. For physical goods, questions of capacity and production cost play important roles in the analysis, whereas capacity and cost are not a concern for digital goods. However, a number of papers in advance selling treat the case of no production costs or capacity constraints, including (Xie and Shugan 2001) and Cachon and Feldman (2017), as special cases, and some papers likes Bhargava and Chen (2012) treat the digital goods case directly. Accordingly, the fact that bonus actions are digital goods are not the main point of departure in our work.

to shut down the spot market. Indeed, we cannot remove the possibility that an uninsured agent needs services in the spot market, and so it is not practical to consider shutting it down. Consider, for instance, a warranty on an engine. Surely, it is not required to buy an extended warranty to have an engine fixed. Indeed, the role of warranties and insurance are *precisely* to avoid high prices in the future for services you may need. Fixing a car or paying for an emergency visit is much less discretionary than buying bonus actions. It is unethical for trauma hospitals to turn someone away just because they do not have medical insurance.

It is unnatural, therefore, in the insurance literature to consider scenarios where the firm is considering shutting down the spot market. Even if a firm wanted to shut down the spot market, they likely could not. When it comes to essential services that insurance typically covers, these are typically not monopoly industries. If a car breaks down, there are often multiple alternatives for where to get it repaired. The commitment to shut down the spot market presumes a tremendous degree of market power. But the question of shutting down the spot market is indeed salient in the case of video games. Here, firms create a virtual world where, by definition, they are monopolists. Bonus actions are not “critical” services. It is credible to commit to shutting down the spot market for such discretionary goods.

It is our deliberation on the question of shutting down the spot market that separates our setting from much of the existing literature on intertemporal pricing, insurance, and warranties. In the case of intertemporal price discrimination literature, the premise is based on continuing sales of a durable good. In the advance sales market, the typical examples are those of shared markets that welcome “late comers” in the spot market and are thus not credible to shut down. In the context of bonus action, firms *can* exclude players from arriving “late” to purchase. The only people who can “see” the spot market are people who had the chance to ”see” the advance sales market (if one was set up). Given this discrepancy, the existing literature does not offer much guidance on questions of shutting down the spot market. Indeed, the default question there is more towards asking if it makes sense to open the *advance* market, given that the spot market is open by default.

Indeed, our results show a high degree of nuance regarding the question of opening or closing the spot market. The tool setting, as opposed to the end good setting, also lends our analysis classifications of games into two types (casual and hardcore games), one where we always shut down the spot market, and the other which depends on the proportion of high and low-skill players in a nonmonotone way. We find these results not only to be new but nontrivial in their dependence on the factors (F1) and (F2). We explain these results in some detail in the pages that follow.

Finally, we want to provide a little context on the background of research in video games, which is a growing area of interest in information systems, operations management, and marketing literature. One significant research direction concerns advertising in games. Turner et al. (2011)

study the deployment of advertising embedded in virtual worlds, while Guo et al. (2019b) and Sheng et al. (2022) study the phenomenon of “rewarded” advertising where players are incentivized to watch advertising with in-game rewards. These rewards are often in virtual currencies whose value is controlled by the game designer, itself a subject of study in recent papers (Guo et al. 2019a, Meng et al. 2021).

Other researchers have studied how available data in video games can be used to study player behavior. Huang et al. (2019), Ascarza et al. (2020) examine how player engagement and retention are impacted by game mechanics (a topic also touched on in Sheng et al. (2022)). Nevskaya and Albuquerque (2019) use video game data to empirically explore the impact of different in-game policies that can limit excessive engagement of players in games, a phenomenon that is concerning to parents and policy-makers.

Among the growing number of papers studying video games, Chen et al. (2021) and Jiao et al. (2021) are most closely related to our paper thematically. Chen et al. (2021) study the design and pricing of “loot boxes”. A loot box contains valuable virtual items and needs to be unlocked using “keys” that are typically sold for real money or in-game virtual currencies. Our research question is similar: we explore the pricing of a video game element (bonus actions in a puzzle game setting), but there are also important differences. Loot boxes serve a mechanic more akin to “collections” in real life, players want to collect and have access to a given array of “weapons” or “clothing” that have varying degrees of value and rarity. By contrast, the bonus actions we study are “consumable” and cannot be meaningfully collected—they are either used for an imminent purpose or lost. This “perishability” gives rise to a different analytical approach. In particular, while Chen et al. (2021) considers a dynamic model for pricing loot boxes for arriving customers, our focus is on a static decision of selling bonus actions to address an imminent potential need. The timing that enters our model concerns the question of differentially pricing bonus actions when sold ahead of this immediate need (that is, “in advance”) or at the time it is needed (that is, “on the spot”).

Jiao et al. (2021) study the selling of virtual items that improve a player’s winning chances, like our bonus actions. They focus on player-versus-player games and investigate ways to induce players to purchase virtual items. Specifically, they examine whether game designers should disclose the opponent’s skill level before the game begins (referred to as a “transparent selling” mechanism to sell virtual items) or conceal this information from players (referred to as an “opaque selling” mechanism). Instead of player-versus-player games, we study puzzle games where direct player interactions are not the emphasis. Moreover, Jiao et al. (2021) assume that the virtual items are sold before the game begins, whereas we let the game designer strategically choose the timing of selling bonus actions.

3. Model basics

A game designer (firm) sells bonus actions to players playing a level of a single-player puzzle game. Firms can sell bonus actions to players *before* they attempt the level (called the advance sales market) and *after* they fail to pass the level (the spot market). The firm must decide on which market to sell bonus actions (advance and/or spot) and the corresponding selling prices. We assume bonus actions are used only *after* a player fails their initial attempt of a level and that there is no second spot market after a second failed attempt. Therefore, players will purchase bonus actions at most once, either in the advance sales market or spot market. We assume that the firm and players are all risk neutral. We also assume the direct cost of providing bonus actions is negligible.

Bonus actions sold in the advance sales market have price p_A. Bonus actions sold in the spot market have price p_S. The price p_S is announced when players fail in their attempt to pass the level.[13] We assume that the price p_S is uniform to all players and thus does not depend on the ending position of an individual player in the puzzle when his/her attempt fails. In other words, a higher price (or lower) p_S is not charged if a player is "closer" to solving the puzzle. Because players can attempt levels repeatedly, and learn from other players what prices they were offered, such price comparisons cause personalized pricing to be viewed as unfair and therefore rare in practice.

If the firm decides not to sell bonus actions in either the advance or spot market, then it must commit to this choice and make it known to the players before they attempt the level. The firm's commitment can easily be verified by the players because players can repeatedly attempt levels in the game and observe the firm's choice. The repeated nature of play in puzzle games allows players to get a good sense of the possible value of p_S in the next attempt. This observation also justifies the use of a rational expectations equilibrium solution concept that we employ below.

If the firm chooses to shut down the spot market and only sell bonus actions in the advance sales market, we call this a *pure advance sales (PAS) strategy*. If the firm chooses to shut down the advance sales market and only sell bonus actions in the spot market, we call this a *pure spot sales (PSS) strategy*. If the firm chooses to offer bonus actions in both markets (with prices that induce positive sales in both markets), we call this a *hybrid advance sales (HAS) strategy*.

3.1. Player and game characteristics

We assume there are two types of players: high-skill players and low-skill players. High-skill players have a higher probability of passing the level than low-skill players. Let β_H denote the probability

[13] In our research looking into games offering bonus actions, the spot price is typically not announced before the start of the level. Indeed, any price announced before the attempt of the level resulting in a fail state is subject to commitment issues. Of course, players who play for a long time come to expect what the spot price will be (we model this as a rational expectations framework below). But this building of expectation is different than the firm declaring a committing to a price *a priori*. For example, in *Happy XiaoXiao Le*, we have seen spot prices discounted from the usual price that players might be accustomed to.

of a high-skill player passing the level without bonus actions and, similarly, β_L for a low-skill player. Naturally, $\beta_L < \beta_H$. The difference $\epsilon = \beta_H - \beta_L > 0$ is a measure of skill heterogeneity among the players. Let N_H be the number of high-skill players attempting the level and N_L denote the number of low-skill players attempting the level. Throughout, we assume that N_H and N_L are both strictly positive. The ratio N_H/N_L plays an important role in our analysis.

If a type i player fails their initial attempt to pass the level, they can use bonus actions to make a second attempt. With bonus actions (if purchased), the player passes the level with probability α_i in the second attempt and fails (a second time) with probability $1-\alpha_i$ ($i = H, L$). One can think of α_i as a random variable that depends on the ending state of the game after a failed attempt, which can be better or worse than the starting position (the parameter α_i is discussed in more detail below). For $i \in \{L, H\}$, we assume that α_i follows a uniform distribution $U[\beta_i - \delta, \beta_i + \delta]$. The parameter δ reflects the entropy of the game (discussed in more detail below).

We assume $\delta > 0$ to avoid a trivial case where bonus actions have no additional value for players to pass the level. In the case where $\delta = 0$, $\alpha_i = \beta_i$. Thus, there is no value in buying the bonus action beyond starting the level again from the beginning, assuming that the player does not experience a time disutility for starting the level over again. Considering that α_i should be between 0 and 1, we further assume $0 \leq \beta_H - \delta < \beta_H + \delta \leq 1$ and $0 \leq \beta_L - \delta < \beta_L + \delta \leq 1$. Type i players are ex-ante homogeneous but ex-interim heterogeneous in the probability α_i. That is, before they attempt, all type i players have the same belief on the distribution on α_i. After a failed attempt, they realize different values for α_i.

A few words on the interpretation of α_i, and why its value may differ from β_i. Players start in a predictable position in the game (that is, the initial condition of the puzzle) while, conditional on not passing, the probability of passing with bonus actions depends on the ending position in the puzzle. This is random and depends on the attempt of the player. One may ask, how is it possible for α_i, on occasion, to be *less than* β_i? In puzzle games, players can certainly end an attempt in a predicament that is *farther* from completion than at the initial position. For instance, in *Candy Crush*, after the player uses her initial allotment of moves, an additional five moves may yield little chance of passing the puzzle if the player squandered her earlier moves. The model assumes that the *expected value* of α_i is β_i, reflecting Martingale-like beliefs about the difficulty for players who purchase in the spot market. In other words, before playing the puzzle, the player expects the difficulty of passing the level with bonus actions from a failed initial attempt to be roughly as hard as passing the level from the beginning with the allotted free actions. The benefit of purchasing the bonus actions *ex-ante* is having an "enhanced" attempt at overcoming this difficulty, and not feeling a psychological loss of *almost* passing the level and having to restart from scratch in a later attempt.

Next, a few words on the parameter δ. This relates to the variance associated with using bonus actions to pass the level. Observe that the *ex-ante* expected probability of passing the level is $\beta_i + (1-\beta_i)\beta_i$ since the expected value of α_i is β_i. Different values for δ give rise to changes in the *variance* of this anticipated passing probability. A puzzle game with a high δ is one where progress in the puzzle is unpredictable and nonlinear. These games may involve random factors or require flashes of "insight" or out-of-the-box thinking to complete. The higher is δ, the more difficult it is to predict the state of the player's progress at the end of an attempt. Whereas, when δ is small, it means that the ending position is easier to predict for the player.

Finally, we consider player payoffs associated with various outcomes. The payoff a player receives for passing a level depends on whether they passed it using bonus actions or not. Let P_N denote the payoff for passing the level on the current attempt without using the bonus action. Let P_B be the payoff of initially failing and using bonus actions to pass the level. We assume that $0 < P_B \leq P_N$ because it can be more satisfying to pass the level without experiencing failure than needing to use bonus actions to pass the level.

Recall that the mobile games we consider are typically "free-to-play", meaning that players can always attempt to pass the level at a later time. Accordingly, P_N and P_B can be seen as payoffs gained for passing the level *now* instead of having to wait to pass the level later (with the possibility of many intermediate failures that waste both time and energy).

We assume that P_N and P_B are uniform across both player types. We assume uniformity in payoffs to accentuate the role of differences in *skill* as the primary driving force of interest. We believe that considering a model that has heterogeneity in both payoffs and skills is an interesting subject, but best kept for future study.

Beyond the payoff of bonus actions for passing the level after a failed attempt, we also model the intrinsic pleasure a player receives for using bonus actions, irrespective of whether the bonus actions help the player pass the level or not. For many games, the use of bonus actions triggers satisfying sounds and images (for example, triumphant music) that make bonus actions intrinsically fun to use. Let v denote this intrinsic valuation of using a bonus action. We assume that v is nonnegative and allow for the possibility that $v = 0$.

In summary, the utility of purchasing bonus actions comes from two sources. The first is from the outcome of using bonus actions to pass the level and earning (with some probability) payoff P_N or P_B. The second is the intrinsic valuation v gained from using the bonus actions. Of course, there is a disutility for purchasing the bonus action, either p_A or p_S, depending on which market it was purchased.

3.2. Player utility

For a type i player, we denote U_i^A as the utility from purchasing bonus actions in the advance sales market, u_i^S as the utility from purchasing bonus actions in the spot market, and U_i^{NA} as the utility from *not* purchasing bonus actions in the advance sales market. The upper case U represents an expected utility before realizing α_i, while the lower case u represents the realized utility after the first attempt and given a realized α_i value. Clearly, the players' utility functions depend on the firm's selling strategy. To illustrate player utility, we take the HAS strategy as an example. This is the most complex case where both advance sales market and spot market are open.

When the firm adopts a HAS strategy, players first decide whether or not to purchase bonus actions in the advance sales market. If not, players attempt to pass the level without bonus actions. If they fail the attempt, players then decide whether or not to purchase bonus actions in the spot market. The sequence of events as well as the corresponding probabilities and payoffs are presented in Figure 1.

The reader will notice in the "no advance purchase" branch of the tree that the choice of p_S is modeled to happen *after* α_i is realized. As discussed earlier in this section, we assume that the firm chooses p_S *uniformly* across all realizations of α_i. The model does use the fact that p_S can be chosen after the firm observes who purchased bonus actions in advance *and* who passed the level on their initial attempt. In other words, we do not model the case where p_S is chosen at the initial stage of the game. This is also reflected in Figure 1.

Following the left-hand branch of the extensive-form game in Figure 1, if a type i player purchases bonus actions in the advance sales market, she expects utility U_i^A that is given by

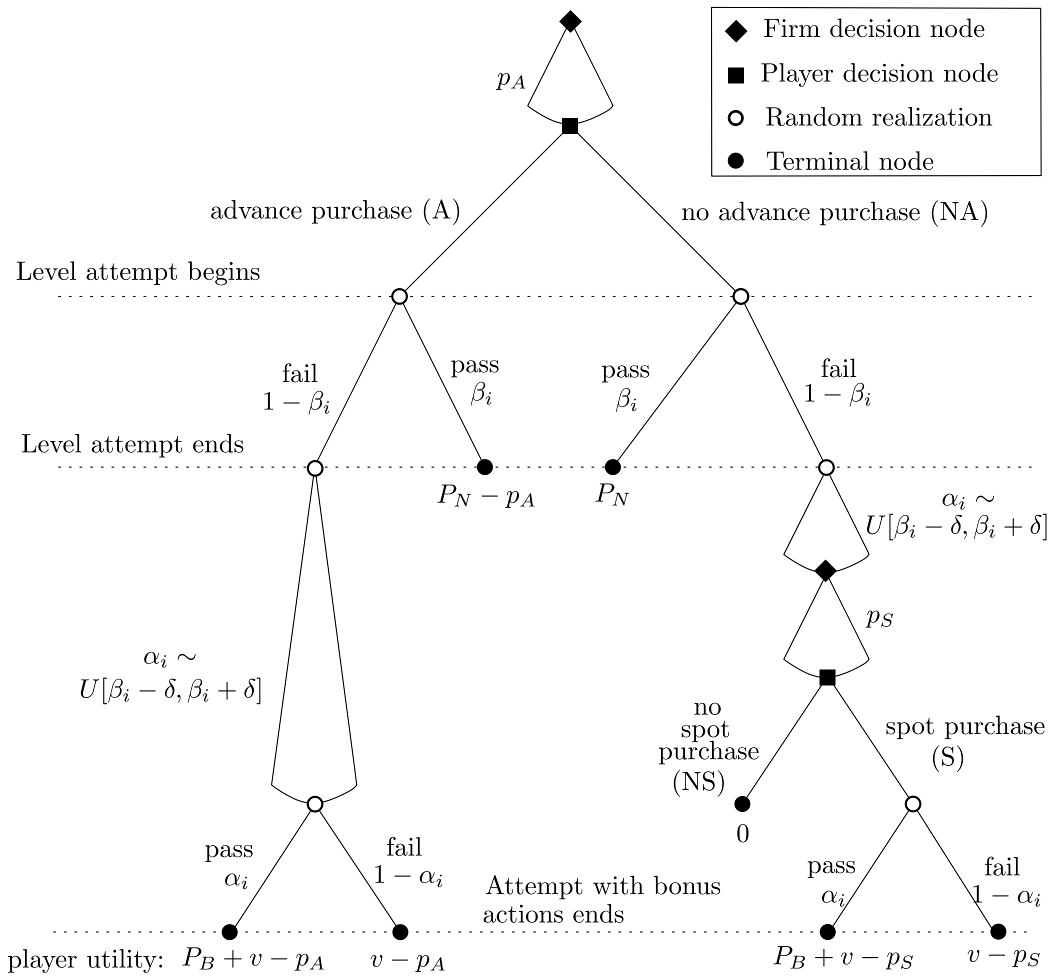
$$\begin{aligned} U_i^A &= \beta_i(P_N - p_A) + (1-\beta_i)\mathbb{E}[\alpha_i P_B + v - p_A] \\ &= \beta_i P_N + (1-\beta_i)(\beta_i P_B + v) - p_A, \qquad \text{for } i = H, L. \end{aligned} \tag{1}$$

If a type i player purchases bonus actions in the spot market, her utility u^S is given by

$$u_i^S = v + \alpha_i P_B - p_S, \qquad \text{for } i = H, L. \tag{2}$$

As seen in Figure 1, we normalize the utility of a player not purchasing bonus actions in the spot market to 0. Thus, a type i player will purchase bonus actions in the spot market if and only if $u_i^S \geq 0$.

Next, we develop the expected utility U_i^{NA} of a type i player *not* purchasing bonus actions in the advance sales market. Under a HAS strategy, players can choose not to purchase in the advance sales market, and wait until the spot market to make a purchase decision (if needed). In order to compute U_i^{NA}, players need to anticipate the spot market price p_S. In our analysis, we use the



p_A = advance-purchase price
p_S = spot purchase price
β_i = prob. of passing w/o bonus actions for a skill i player
α_i = prob. of passing w/ bonus actions for a skill i player
P_N = utility of passing the level without bonus actions
P_B = utility of passing the level with bonus actions
v = utility of using purchased bonus actions

Figure 1 Description of the timeline, decisions, and player payoffs of the hybrid model.

rational expectations (RE) equilibrium of the game, building on Coase (1972) notion that players will understand that the firm will adopt the spot price that maximizes spot profits. The notion is well-justified here because players can repeatedly attempt levels in a puzzle game and so get a good idea of the firm's optimal choice of p_S through experience. The concept of rational expectations equilibrium has been widely adopted in operations management and marketing literature (see, e.g., Li and Zhang (2013), Xie and Shugan (2001), and references therein).

Following the right-hand branch of the extensive-form game in Figure 1, if a type i player decides *not* to buy in advance, she will purchase bonus actions in the spot market only when she fails the initial attempt and $u_i^S \geq 0$. Therefore, her utility U_i^{NA} for *not* purchasing bonus actions in advance can be computed as below:

$$U_i^{NA} = \beta_i P_N + (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - \hat{p}_S)^+], \quad \text{for } i = H, L. \tag{3}$$

We use the notation $[A]^+ := \max\{A, 0\}$, $[A]^- := \max\{-A, 0\}$ and note that $A = [A]^+ - [A]^-$.

Observe that the player must form a belief $\hat{p}_S$ of what the firm will price bonus actions in the spot market in order to make its initial decision of whether to make an advance purchase or not. A rational player will expect that the firm will set a spot price to maximize the spot market profit. In an RE equilibrium, the belief $\hat{p}_S$ matches the firm's actual choice of p_S. In other words, $\hat{p}_S = p_S$. Our analysis assumes an RE equilibrium throughout and so we will drop the notation $\hat{p}_S$ in favor of simply writing p_S in the player's decision problems.

Lastly, we remark that if the firm adopts a PSS strategy and commits to selling bonus actions only in the spot market, then U_i^A and U_i^{NA} are meaningless. Player utility u_i^S from purchasing bonus actions in the spot market is the same as (2). If the firm adopts a PAS strategy and commits to selling bonus actions only in the advance sales market, then u_i^s becomes meaningless. Players' utility U_i^A from purchasing bonus actions in the advance sales market will be the same as (1) whereas their utility U_i^{NA} from *not* purchasing bonus actions in the advance sales market will be $U_i^{NA} = \beta_i P_N$.[14]

4. Player's decision

Players decide whether or not they purchase bonus actions, and if both advance sales and spot markets are open, in which market they purchase bonus actions. Suppose the advance sales market is open. This happens when the firm adopts a PAS strategy or a HAS strategy. A type i player will purchase bonus actions in the advance sales market if and only if $U_i^A \geq U_i^{NA}$ and $U_i^A \geq 0$. The constraint $U_i^A \geq U_i^{NA}$ is an incentive compatibility (IC) constraint. The constraint $U_i^A \geq 0$ is an individual rationality (IR) constraint. Note that $U_i^{NA} = \beta_i P_N > 0$ under a PAS strategy and $U_i^{NA} = \beta_i P_N + (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+] > 0$ under a HAS strategy. Therefore, the IR constraint $U_i^A \geq 0$ is implied by the IC constraint.

Suppose the spot market is open. This happens when the firm adopts a PSS strategy or a HAS strategy. As discussed in Section 3.2, a type i player will purchase bonus actions in the spot market (if needed) if and only if bonus actions result in a non-negative utility, i.e., $u_i^S = v + \alpha_i P_B - p_S \geq 0$. That is, only those players with a sufficiently high probability α_i of passing the level with bonus actions will buy them.

[14] The description of the sequence of events under the PAS and PSS strategies follow a similar pattern and not detailed explicitly here.

Lemma 1 *(a) Under a PAS strategy, a player of type i will purchase bonus actions (before the attempt) if and only if* $p_A \le (1-\beta_i)(v+\beta_i P_B)$.

(b) Under a PSS strategy, a player of type i will purchase bonus actions (after failing the attempt) if and only if $p_S \le v + \alpha_i P_B$.

(c) Under a HAS strategy, a player of type i will purchase bonus actions before the attempt if and only if $p_A \le (1-\beta_i)\{(\beta_i P_B + v) - \mathbb{E}[(\alpha_i P_B + v - p_S)^+]\}$. *For those players who choose not to purchase in the advance sales market, they will purchase bonus actions in the spot market (if needed) if and only if* $p_S \le v + \alpha_i P_B$.

We pay particular interest in a HAS strategy where the firm chooses to offer bonus actions in both markets and set prices that induce positive sales in both markets (more details on this in Section 5). As we have two types of players (high- and low-skill), under a HAS strategy there exist two possible scenarios: (a) high-skill players consider buying in the spot market while low-skill players consider buying early, and (b) low-skill players consider buying in the spot market while high-skill players consider buying early. To highlight the fundamental difference between the two scenarios, we call (a) a *regular HAS strategy* and (b) a *reverse HAS strategy.*

Remark 1 *It is important to stress that the pure advance strategy and the pure spot strategy are not special cases of a HAS strategy (either the regular or reverse). One might think this given that a HAS strategy is associated with opening up both markets and offering a price in each, and so one could set a sufficiently high price under a HAS strategy to effectively "shut down" one market or the other. However, such a strategy is* not *a HAS strategy by our definition. As mentioned at the beginning of the last paragraph (and as will be seen in later development), hybrid selling strategies are those where prices are set to induce positive sales in both markets.*

Below, we carefully explore the utility difference $U_i^A - U_i^{NA}$ under a HAS strategy that serves an important role in determining which players will purchase in the advance sales market. Some algebra produces the following description of the utility difference $U_i^A - U_i^{NA}$ from Equations (1) and (3) describe the net value for an advance purchase:

$$
\begin{aligned}
U_i^A - U_i^{NA} &= (1-\beta_i)(\beta_i P_B + v) - p_A - (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+] \\
&= \underbrace{(1-\beta_i)(p_S - p_A)}_{\text{price discount}} - \underbrace{\beta_i p_A}_{\text{waste of bonus actions}} - \underbrace{(1-\beta_i)\mathbb{E}[(v + \alpha_i P_B - p_S)^-]}_{\text{potential negative surplus}}. \qquad (4)
\end{aligned}
$$

The first term measures the benefit of buying early, that is a price discount for purchasing bonus actions in advance; namely, it is the product of the markup $p_S - p_A$ in the spot market weighted by the probability $1-\beta_i$ that a purchase is even needed in the spot market. The second term is a loss associated with buying bonus actions in advance that is not used. This happens with probability β_i. It is straightforward to see the first term decreasing in β_i and the second term increasing in β_i.

The third term is the loss associated with buying bonus actions in advance that is actually used, i.e., the player failed the level at the first attempt. At the end of the first attempt, if a player's realized α_i is very small—meaning that her second chance at passing the level using bonus actions is low—then purchasing bonus actions may result in a negative surplus or loss. The player will not purchase the bonus actions in the spot market. However, the same player might have made an early purchase of bonus actions. In this scenario, the player incurs a loss associated with advance buying, which is captured in the third term. Given that α_i follows a uniform distribution, $U[\beta_i - \delta, \beta_i + \delta]$, we can easily show that the third term decreases in β_i. As a result, the difference $U_i^A - U_i^{NA}$ may not be monotone in β_i.

Remark 2 *We want to highlight how our analysis of (4) is a significant departure from the extant advance selling literature focusing on end goods. Because end goods do not have so many uncertainty layers as tools, the third term in (4) degenerates to a simple constant, e.g., $\beta_i L_i$ in Bhargava and Chen (2012) and βL in Xie and Shugan (2001) (note that the notation borrows some from our paper to allow for more ready comparison), which is dominated by the second term. As a result, the difference $U_i^A - U_i^{NA}$ is monotone in β_i in end goods advance selling literature, meaning that only regular hybrid is possible.*

The possibility of having both hybrid and reverse hybrid strategies is illustrated concretely in the following example.

Example 1 *Consider the instance with $v = 1$, $P_N = 2$, $P_B = 5$, $\delta = 0.1$, and consider the given prices $p_A = 1.5$ and $p_S = 3$. Table 1 (with $\beta_H = 0.3$ and $\beta_L = 0.1$) displays the scenario that high-skill players prefer to buy early but low-skill players prefer to buy at spot. Table 2 (with $\beta_H = 0.6$ and $\beta_L = 0.4$) displays the scenario that low-skill players prefer to buy early but high-skill players prefer to buy at spot.*

	type H player	type L player
discount: $(1-\beta_i)(p_S - p_A)$	1.05	1.35
waste: $\beta_i p_A$	0.45	0.15
potential negative surplus: $(1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^-]$	0.35	1.35
$U_i^A = \beta_i P_N + (1-\beta_i)(\beta_i P_B + v) - p_A$	0.85	0.05
$U_i^{NA} = \beta_i P_N + (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+]$	0.6	0.2
$U_i^A - U_i^{NA}$ = discount - waste - potential negative surplus	0.25	-0.15

Table 1 **Example where $U_H^A - U_H^{NA} > 0 > U_L^A - U_L^{NA}$. (Assume $\beta_H = 0.3$ and $\beta_L = 0.1$)**

	type H player	type L player
discount: $(1-\beta_i)(p_S - p_A)$	0.6	0.9
waste: $\beta_i p_A$	0.9	0.6
potential negative surplus: $(1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^-]$	0	0.075
$U_i^A = \beta_i P_N + (1-\beta_i)(\beta_i P_B + v) - p_A$	1.3	1.1
$U_i^{NA} = \beta_i P_N + (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+]$	1.6	0.875
$U_i^A - U_i^{NA} =$ discount - waste - potential negative surplus	-0.3	0.225

Table 2 Example where $U_H^A - U_H^{NA} < 0 < U_L^A - U_L^{NA}$. (Assume $\beta_H = 0.6$ and $\beta_L = 0.4$)

As we can see, although low-skill players enjoy a higher discount for buying early and lower waste, their potential negative surplus associated with buying bonus actions in the spot market is also higher. Hence, it is unclear which type of player has a higher net value for an advance purchase.

When the "residual" uncertainty α_i is considered, players with *less* skill may choose to wait to purchase bonus actions while more skilled players purchase bonus actions in advance. This is due to the possibility that bonus actions can be priced in such a way that only "lucky" low-skill players who make better-than-average progress towards passing the level will find bonus actions valuable enough to purchase *ex-interim*, but this price is too high for skilled players, who prefer to buy *ex-ante* at a discounted price. The low-skilled player's expected value of bonus actions can be lower than the discount price *ex-ante*, but a portion of low-skill players facing different residual uncertainties may find bonus actions sufficiently valuable *ex interim* to warrant a purchase. The important factor here is that the residual uncertainty inherent in using bonus actions can induce a wide range of expected values for players of differing skills.

Remark 3 *In this remark, we further expand on the distinction between our paper and that of Bhargava and Chen (2012) (and related literature). To do so, we must make clear another difference that arises in the tool context that differs from the end good context in terms of the classification of customer types. Our customer segments, based on notions of high and low skill, do not align with the "mass" and "niche" customer categories discussed in Bhargava and Chen (2012). The low-skill type is one who is more likely to need the tool but, interestingly, often finds it less useful at the time of usage. A low-skill player is more likely to fail the level (since β_L is smaller than β_H), but is also more likely to fail the level even with the benefit of using bonus actions (α_L is more likely to yield a smaller result than α_H). These two types do not have a direct mapping to "niche" and "mass". Indeed, there may be some games where the mass of players is low-skill, whereas, in others, the mass of players is high-skill. Indeed, some games are designed to be "inviting" to newer, less-skilled gamers, while others court experienced players.*

The possibility of both the hybrid and reverse hybrid strategies presents challenges in our analysis, and we will proceed by first analyzing a case that rules out this complexity. The following lemma helps us identify such a case.

Lemma 2 *Suppose $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$. Under a HAS strategy, we have $U_L^A - U_L^{NA} \geq U_H^A - U_H^{NA}$ for any p_A and p_S. That is, it will never transpire that high-skill players buy in advance and low-skill players buy in the spot no matter the choice of p_A and p_S of the firm.*

Lemma 2 implies that when β_L is sufficiently high, low-skill players are always more motivated to buy early than high-skill players. In other words, the firm can never set prices to induce high-skill players to buy in advance and low-skill players to buy in the spot. In this setting, a HAS strategy must be a regular HAS strategy.

In the proof of Lemma 2, we actually show that the condition $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$ is a sufficient and necessary condition. If $\beta_L < (1-\beta_H) - \frac{v}{P_B}$, the firm can find prices p_A and p_S that induce high-skill players to buy in the spot and low-skill players to buy in advance. Namely, a reverse HAS strategy may be feasible. Furthermore, we find that the same condition $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$ has implications for the PAS strategy.

Corollary 1 *Under a pure advance selling strategy, if $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$, we have $U_L^A - U_L^{NA} \geq U_H^A - U_H^{NA}$ for any p_A. If $\beta_L < (1-\beta_H) - \frac{v}{P_B}$, we have $U_L^A - U_L^{NA} < U_H^A - U_H^{NA}$ for any p_A.*

Corollary 1 suggests that as long as β_L is sufficiently high, even if the spot market is not available and the firm commits to selling bonus actions only in the advance sales market, low-skill players are more likely to buy early than high-skill players. But if β_L is relatively low, high-skill players become more likely to buy early than low-skill players.

Motivated by Lemma 2 (and Corollary 1), we classify games into two types. We call games with relatively high β_L (i.e., $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$) *casual games* and games with relatively low β_L (i.e., $\beta_L < (1-\beta_H) - \frac{v}{P_B}$) *hardcore games*. We will characterize the optimal selling strategies for casual and hardcore games in Sections 6 and 7 respectively.

Examples of casual games are those marketed to a mass audience that start with an easy learning curve that encourages many people to play. Examples include *Candy Crush*, *Cute the Rope*, and *Words with Friends*. In some of these games, the difficulty is adapted to the player's skill level by matching players in competitive settings with similar skill levels. Even for more difficult games, the initial levels may be easier to progress through, making the games more casual initially (we return to this theme in later discussions). Examples of more challenging puzzle games are *Red Puzzle Game* and *Beat Stomper*, which require outside-of-the-box thinking and punishingly accurate hand-eye coordination, respectively. These games are known for their challenge. An inexperienced player is very unlikely to make it far in these games, suggesting that β_L is sufficiently low to be classified as hardcore games in our framework.

We end this section with a couple of comments about the bound $\beta_L \geq (1-\beta_H) - v/P_B$, which plays an important role in our paper. First, note that the right-hand side of this inequality is less than 1, but could be negative. This implies that it is possible for a given set of parameters, that all values of β_L would get classified as a casual game. Also, when $v=0$, the bound yields a clean interpretation: a game is casual if the success probability of a low-skill player exceeds the failure probability $(1-\beta_H)$ of a high-skill player.

5. Firm's decision

As detailed in the previous two sections, the firm has four selling strategies—PAS, PSS, regular HAS, and reverse HAS. In order to find the optimal selling strategy, the firm optimizes the prices under each selling strategy, and from among these chooses the strategy that optimizes revenue.

In this section, we describe the firm's optimization problem under each of the four selling strategies. For brevity, the optimal prices and revenue under each selling strategy are characterized in the appendix. We denote the optimal revenue for the four selling strategies Π^A (pure advance), Π^S (pure spot), Π^H (regular hybrid), and Π^{RH} (reverse hybrid). The firm's revenue under the optimal selling strategy is denoted Π^*, which satisfies $\Pi^*=\max\{\Pi^A,\Pi^S,\Pi^H,\Pi^{RH}\}$.

5.1. Firm adopts a PSS strategy

Here, the firm shuts down the advance sale market and sells bonus actions only in the spot market. It chooses the price p_S for bonus actions to maximize its revenue. The firm's optimization problem is

$$\max_{p_S\geq 0}\Pi(p_S):=p_S\left\{N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B-p_S\geq 0)]+N_L(1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B-p_S\geq 0)]\right\},$$

where the form of the profit function Π in this expression comes from the following logic. Lemma 1 indicates that players will purchase bonus actions at price p_S only when they fail their initial attempt and have a sufficiently high α_i. Accordingly, $N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B-p_S\geq 0)]$ is the expected number of high-skill players who will purchase bonus actions and $N_L(1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B-p_S\geq 0)]$ is the expected number of low-skill players who will purchase bonus actions, where $\mathbb{E}[\cdot]$ is the expectation over the distribution of α_i and $i=H,L$ (respectively) and $\mathbb{1}(\cdot)$ is the indicator function. The fact that the firm is risk neutral and the price p_S is set uniformly across all α_i justifies taking expectations over α_i in the computation.

In equations (A.2) and (A.4) of the appendix, we show that $\Pi(p_S)$ is a piecewise continuous function, but it may not be unimodal. Nevertheless, each piece of $\Pi(p_S)$ is either linear or quadratic in p_S. Using this insight, we characterize the optimal price to be at a kink point or satisfy the first-order condition. See Lemma A.3 in the appendix.

5.2. Firm adopts a PAS strategy

Here, the firm shuts down the spot market and commits selling bonus actions only before the attempt. If a player chooses not to buy early, she will not have a second chance of buying bonus actions if she fails the attempt. As discussed in Section 4, a type i player will purchase bonus actions at price p_A if and only if $U_i^A = \beta_i P_N + (1-\beta_i)(\beta_i P_B + v) - p_A \geq U_i^{NA} = \beta_i P_N$, or equivalently, $p_A \leq (1-\beta_i)(\beta_i P_B + v)$.

The firm determines the price p_A for bonus actions to maximize its revenue. Following Corollary 1, we know that $(1-\beta_H)(\beta_H P_B + v) \leq (1-\beta_L)(\beta_L P_B + v)$ if $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$, whereas $(1-\beta_H)(\beta_H P_B + v) > (1-\beta_L)(\beta_L P_B + v)$ if $\beta_L < (1-\beta_H) - \frac{v}{P_B}$. As a result, the firm's revenue will be different for casual and hardcore games.

For casual games, a larger price discount is needed to motivate high-skill players to buy in advance, in comparison to low-skill players. In this case, the firm's optimization problem is given by

$$\max_{p_A \geq 0} \Pi(p_A) := \begin{cases} p_A(N_H + N_L), & \text{if } p_A \leq (1-\beta_H)(\beta_H P_B + v), \\ p_A N_L, & \text{if } (1-\beta_H)(\beta_H P_B + v) < p_A \leq (1-\beta_L)(\beta_L P_B + v), \\ 0, & \text{if } p_A > (1-\beta_L)(\beta_L P_B + v). \end{cases}$$

For hardcore games, a larger price discount is needed to motivate low-skill players to buy in advance, in comparison to high-skill players. Thus, the firm's optimization problem is given by

$$\max_{p_A \geq 0} \Pi(p_A) := \begin{cases} p_A(N_H + N_L), & \text{if } p_A \leq (1-\beta_L)(\beta_L P_B + v), \\ p_A N_H, & \text{if } (1-\beta_L)(\beta_L P_B + v) < p_A \leq (1-\beta_H)(\beta_H P_B + v), \\ 0, & \text{if } p_A > (1-\beta_H)(\beta_H P_B + v). \end{cases}$$

In both cases, $\Pi(p_A)$ is piecewise linear but not continuous. Therefore, the optimal price must be at one of the breakpoints, either $(1-\beta_L)(\beta_L P_B + v)$ or $(1-\beta_H)(\beta_H P_B + v)$. If p_A is chosen in a PAS strategy to target both high-skill and low-skill players we call this a PAS-HL strategy. If p_A is chosen in a PAS strategy to target only low-skill players, we call this a PAS-L strategy. A PAS-H strategy is similarly defined.

5.3. Firm adopts a regular HAS strategy.

Here, the firm sells bonus actions in both the advance sales and spot markets and sets prices p_A and p_S that induce high-skill players to make purchases in the spot market and low-skill players to make purchases in the advance sales market.

Since we assume that the firm determines and announces the prices dynamically, we analyze the optimization problem backwards. First, in the spot market, the firm determines the price p_S to maximize its spot market revenue Π_S. Since a regular HAS strategy restricts attention to the case that high-skill players purchase in the spot market, the firm's optimization problem in the spot market is given by

$$\max_{p_S \geq 0} \Pi_S(p_S) := p_S N_H (1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B - p_S \geq 0)]. \tag{5}$$

Given the optimal spot price p_S^*, the firm chooses p_A to maximize its revenue from low-skill players in the advance sales market. Let Π_A denote the firm's revenue in the advance sales market. The resulting optimization problem is

$$\begin{aligned} &\max_{p_A \geq 0} \Pi_A(p_A) := p_A N_L \\ &\quad \text{s.t.} \quad p_A \leq (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\} \quad (6) \\ &\qquad\qquad p_A > (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\}. \quad (7) \end{aligned}$$

Following Lemma 1, Constraints (6) and (7) ensure that, after observing the price p_A and anticipating the spot price p_S^*, low-skill players will choose to purchase bonus actions in advance and high-skill players will choose to purchase bonus actions in the spot market. These conditions ensure that a positive number of bonus actions is chosen in each market. This confirms what was discussed in Remark 1 above regarding the definition of the HAS strategy.

If there exists a price p_A that satisfies constraints (6)–(7) for some p_S^* solving (5), we say that *an optimal regular HAS strategy exists*. For casual games, the existence of the optimal regular HAS strategy is guaranteed by Lemma 2. However, for hardcore games, it is possible that given the optimal spot price p_S^*, we cannot find any price p_A satisfying constraints (6) and (7). That is, for hardcore games, the optimal regular HAS strategy may not exist, and in this case, we will simply set $\Pi^H = 0$.

5.4. Firm adopts a reverse HAS strategy

Here, the firm sells bonus actions in both advance sale and spot markets and sets prices p_A and p_S to induce low-skill players to make purchases in the spot market and high-skill players to make purchases in the advance sales market.

Similar to Section 5.3, we first solve the firm's problem in the spot market

$$\max_{p_S \geq 0} \Pi_S(p_S) := p_S N_L (1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B - p_S \geq 0)]. \tag{8}$$

Given the optimal spot price p_S^*, the firm chooses p_A to maximize its revenue from high-skill players in the advance sales market. Thus, the firm solves the following optimization problem:

$$\begin{aligned} &\max_{p_A \geq 0} \Pi_A(p_A) := p_A N_H \\ &\quad \text{s.t.} \quad p_A > (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\} \quad (9) \end{aligned}$$

$$p_A \leq (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\}. \tag{10}$$

Constraints (9) and (10) guarantee that given the prices p_A and p_S^*, high-skill players will choose to purchase bonus actions in advance and low-skill players will choose to purchase bonus actions in the spot market. These conditions ensure that a positive amount of bonus actions are chosen in each market. Again, this confirms what was discussed in Remark 1 above.

If there exists a price p_A that satisfies constraints (9)–(10) for some p_S^* solving (8), we say that *an optimal reverse HAS strategy exists*. Lemma 2 implies that the optimal reverse HAS strategy does not exist for casual games. In this case, we set $\Pi^{RH} = 0$. For hardcore games, given the optimal spot price p_S^*, we may be able to find a price p_A satisfying Constraints (9) and (10). That is, for hardcore games, the optimal reverse HAS strategy may exist. Example 1 illustrates such a situation.

Below, we characterize the optimal selling strategy for casual games (Section 6) and for hardcore games (Section 7). As mentioned earlier, the firm optimizes the prices under each candidate selling strategy, and from among these, chooses the one with the highest revenue. Therefore, in the following discussion, whenever we say "the PAS strategy" or "the regular HAS strategy", we refer to those under optimal prices.

6. Casual games

In this section, we consider the case of casual games—first defined at the end of Section 4—where β_L is sufficiently high, meaning that low-skill players have a high probability of passing the level without bonus actions. Specifically, we assume the following throughout Section 6.

Assumption 1 (Casual game) $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$.

We would like to characterize the optimal selling strategy for casual games. Following the discussion in Section 5, we know that the reverse HAS strategy does not exist for casual games. To find the optimal selling strategy, we compare the firm's optimal revenues under the PAS strategy, the PSS strategy, and the regular HAS strategy. That is, we compare Π^A, Π^S and Π^H, and find the one with the largest revenue.

Our first result states that the regular HAS strategy is always better than the PSS strategy.

Proposition 1 *For casual games, the regular HAS strategy dominates the PSS strategy. That is,* $\Pi^H > \Pi^S$.[15]

[15] Equality holds only if one of N_H or N_L is zero or $\beta_H = \beta_L$. These are cases that we exclude in our model, as discussed in Section 3.

Remark 4 *At first glance, Proposition 1 may seem entirely expected because one may think that PSS is just a special case of HAS by setting the advance sale price p_A to be sufficiently large under HAS. However, this is not the case because PSS is* not *a special case of HAS. Recall Remark 1 highlights that the definition of a regular HAS strategy is to have positive sales amounts in both markets and constraint the choices of p_A and p_S accordingly (see Section 5.3 for details.)*

According to Proposition 1, the optimal selling strategy for casual games should be either the PAS strategy or the regular HAS strategy. We further examine when one of the two strategies dominates in the following theorem. This gives us insight into the decision of when to close the spot market, which has not been explored in the previous literature (as detailed in Section 2).

Theorem 1 *For casual games, there exist two (non-negative) thresholds, $\underline{n}$ and $\bar{n}$, for the ratio N_H/N_L.*

- *When $N_H/N_L \leq \underline{n}$ or $N_H/N_L \geq \bar{n}$, it is optimal to shut down the spot market and pursue the PAS strategy. That is, $\Pi^A \geq \Pi^H$.*
- *When $\underline{n} < N_H/N_L < \bar{n}$, it is optimal to pursue the regular HAS strategy. That is, $\Pi^A < \Pi^H$.*

Theorem 1 indicates that only when the market is balanced between low-skill players and high-skill players, the regular HAS strategy is optimal. Otherwise, the PAS strategy is optimal. The characterization of the two thresholds $\underline{n}$ and $\bar{n}$ are provided in equations (A.8) and (A.9) of the appendix.

At a high level, this result balances two important forces. On the one hand, there is the power of having two markets and the ability to price discriminate between these two markets. On the other hand, with PAS, there is the value of the firm committing to shutting down the spot market, which can motivate players to purchase early by removing any potential utility for waiting. It is not surprising that there are scenarios where one of these two benefits dominates over the other depending on the parameters of the model.

It is less expected, however, that the resulting relationship is not monotone in the proportion of skilled players. Theorem 1 indicates that the PAS strategy is optimal when there are relatively few high-skill players ($N_H/N_L \leq \underline{n}$) or a high proportion of high-skill players ($N_H/N_L \geq \bar{n}$). But when the proportion of high-skill players is moderate, the regular HAS strategy becomes optimal. This non-monotonicity in the proportion of N_H/N_L can be explained by the existence of the two regimes of the optimal PAS strategy—PAS-L and PAS-HL (defined in Section 5.2)—and the fact that the optimal regular HAS strategy does not change its structure as N_H/N_L changes (following the derivations of PAS and HAS in the appendix).

The intuition is as follows. The PAS-L strategy is optimal when there are very few high-skill players ($N_H/N_L \leq \underline{n}$). In casual games, low-skill players are more likely to buy early than high-skill players (according to Lemma 2 and Corollary 1). When the firm charges the highest advance sales price that the low-skill players are willing to pay, low-skill players purchase, but high-skill players do not. If the firm wants to attract high-skill players to buy, it has to further lower the price in the advance sales market or open the spot market. However, considering that there are relatively few high-skill players, the increased sales from high-skill players cannot justify the profit margin loss from low-skill players. Thus, the firm should only serve low-skill players and stay committed to closing the spot market.

For intermediate proportions of high- and low-skill players, it is optimal to follow a HAS strategy. The spot price can be set to attract high-skill players but not significantly impact the price for bonus actions sold to low-skill players in advance. Here we see the benefits of price discrimination. The price in the advance sales market can stay sufficiently high since it does not need to attract high-skill players. This allows for a proportion of high-skill players to realize sufficiently small values of α_i to warrant purchases in the spot market.

However, as the proportion of high-skill players increases, the firm adopts the PAS-HL strategy. Although a lower price is needed to attract high-skill players to buy in advance rather than in the spot market, it can be sold to a larger proportion of them. Indeed, high-skill players are homogeneous before α_i is realized, so a price can be chosen so that all high-skill players purchase early. Of course, this is a lower price than would be needed to sell only to low-skill players, but now there are sufficiently many high-skill players to justify the lower price. The fact that there is no spot market, captures value in the advance sales market from high-skill players who would otherwise wait to see if they needed to buy bonus actions in the impending spot market.

Theorem 1 provides implications for selling bonus actions in practice. When a game is initially introduced to the market, almost everyone is playing for the first time, so they are likely to be low-skill players. As players play the game, some of them become high-skill players, and the proportion of high-skill to low-skill players increases. Eventually, as the game enters a maturing stage, the majority of players are experienced high-skill players because only “die-hard” fans stick with the game, and new adoptions of the game are less frequent. Thus, our result suggests that, throughout the life-cycle of a game, the firm should start with a *PAS strategy*, then adopt a *HAS strategy*, finally switching back to a *PAS strategy.*

Theorem 1 further suggests the firm should adopt different selling strategies for different levels of the game. Usually, a level-based puzzle game starts with easy puzzles that attract many low-skill players. As players progress through the levels of the game, the proportion of high-skill to low-skill players increases. This could be because later puzzles are more challenging and low-skill players have difficulty in advancing to these levels. It is also possible that low-skill players evolve into high-skill players as they ascend to higher levels. Thus, our findings suggest that bonus actions should only be sold in advance at early levels. A HAS strategy is preferred at intermediate levels where low-skill players start to drop-off. Finally, the firm would return to PAS strategy as mostly only experienced players remain.

The puzzle game *Happy XiaoXiao Le* follows a HAS strategy, suggesting, according to our findings, that the game has a mix of high-skill and low-skill players. This is consistent with the fact that many puzzle games are designed to be attractive to a wide variety of players with differing levels of skill. Yet, it is worth noting that, at the time of writing this paper, *Candy Crush* did not offer advance sales of their popular "five extra moves" bonus actions; they are only offered in the spot market. This, however, need not contradict our theory. We discuss this in more detail following the statement of Proposition 4 below.[16]

7. Hardcore games

In this section, we study the optimal selling strategy for hardcore games with relatively low β_L. We make the following assumption throughout Section 7.

Assumption 2 (Hardcore game) $\beta_L < (1-\beta_H) - \frac{v}{P_B}$.

In Section 4, we have described how both the optimal regular hybrid and the optimal reverse hybrid can both become feasible in hardcore games. This feasibility needs to be handled carefully when analyzing hardcore games. The detailed conditions for the existence of the optimal regular HAS strategy and the optimal reverse HAS strategy are provided in Lemma A.4 and Lemma A.6, respectively, in the appendix.

As a result, for hardcore games, all four selling strategies are candidates for the optimal strategy. To find the optimal selling strategy, we compare the firm's optimal revenues under an optimal PAS strategy (Π^A), an optimal PSS strategy (Π^S), an optimal regular HAS strategy (Π^H, if one exists), and an optimal reverse HAS strategy (Π^{RH}, if one exists). We prove that the PAS strategy dominates all other strategies for hardcore games.

Theorem 2 *For hardcore games, the optimal selling strategy is to shut down the spot market and adopt the PAS strategy. That is,* $\Pi^A \geq \Pi^S$, $\Pi^A \geq \Pi^H$, *and* $\Pi^A \geq \Pi^{RH}$.

Theorem 2 indicates that the firm should always commit to selling bonus actions in advance and shut down the spot market for hardcore games. Removing sales in the spot market allows the firm

[16] There is also one complication on the Apple platform for games. The minimum payment on the platform is \$0.99 USD. This restriction can limit the implementability of a PAS strategy, where optimal prices may fall below that range. An interesting extension of our model here might study the implication of restricting prices to a "price ladder". Price ladders have been studied with some interest in the revenue management literature (see, for instance, Sumida et al. (2021)).

to charge a higher price to more players in the advance sales market, thus benefiting the firm. If opened, the spot market can become crowded by low-skill players because it is difficult for these players to pass the level. However, these low-skill players do not value the bonus actions very highly, because they cannot easily pass the level even with additional help. Selling to low-skill players in the spot market makes the spot price too low. Indeed, because the bonus actions are relatively "weak" on average for low-skill players (because β_L is low), a low-skill type player has an incentive to wait to see if they get lucky and *almost* finish the level before buying bonus actions. So the waiting incentive is high when β_L is small. Cutting the spot market cuts out this speculation and allows for a higher advance sale price, driving up revenue. It can, therefore, be more profitable to commit to shutting off the spot market.

8. Discussion

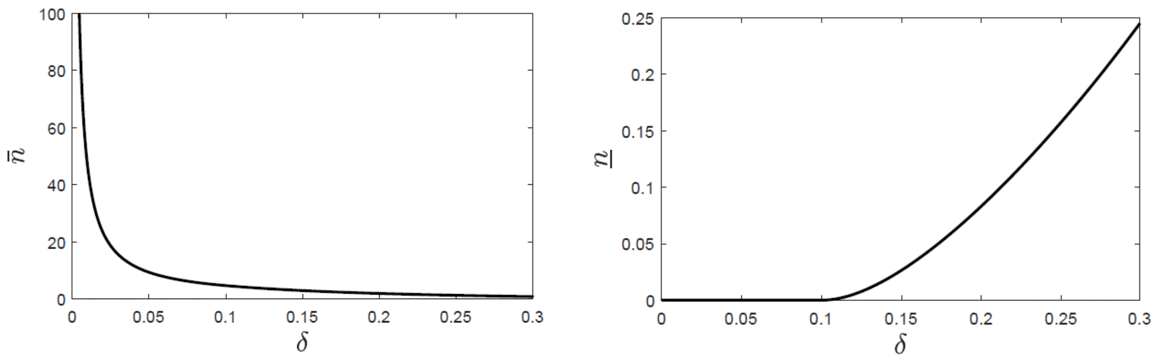
We now discuss how the optimal selling strategy is impacted by game characteristics (Section 8.1) and market characteristics (Section 8.2). This will allow us to answer (Q1) and (Q2) in light of (F1) and (F2), first raised in the introduction. This discussion will focus on casual games for which the optimal strategy could be either the PAS strategy or the regular HAS strategy. Indeed, for hardcore games, the PAS strategy is always optimal. Then in Section 8.4, we explore the total player welfare under both casual and hardcore games.

8.1. Impact of game characteristics

Recall that entropy δ measures the predictability of an attempt's progress. As mentioned before, games with high δ are those with significant random components where the ending position is hard to predict for the player. We refer to such settings as *games of chance*. In contrast, games with low δ are referred to as *games of skill*. The ending position of these games is easier to predict for the player. Entropy, to some extent, can be controlled by the firm. For example, when designing a level, the publisher can add or remove random elements.

We now explore how a level's entropy has an impact on selling strategies. Theorem 1 indicates that the firm should adopt the HAS strategy only when the ratio N_H/N_L is intermediate; that is, $\underline{n} < N_H/N_L < \bar{n}$. In Figure 2, we plot the two thresholds, $\underline{n}$ and $\bar{n}$, as functions of δ for a given instance. Observe that the lower threshold $\underline{n}$ increases in δ whereas the upper threshold $\bar{n}$ decreases in δ, suggesting that for games with large entropy, the HAS strategy becomes less attractive. In other words, the firm adopts the PAS strategy for a wider range of parameters. This is formally established in the following proposition.

Proposition 2 *Recall the upper and lower thresholds $\bar{n}$ and $\underline{n}$ defined in Theorem 1 for casual games. The upper threshold $\bar{n}$ (when positive) decreases in δ while the lower threshold $\underline{n}$ (when*



(a) upper threshold $\bar{n}$ (b) lower threshold $\underline{n}$

Figure 2 The upper and lower thresholds change with δ. (Fix $\beta_H = 0.7, \beta_L = 0.5, v = 1$ and $P_B = 2$)

positive) increases in δ. In other words, as level entropy δ increases, the firm is more likely to adopt the PAS strategy than the HAS strategy.

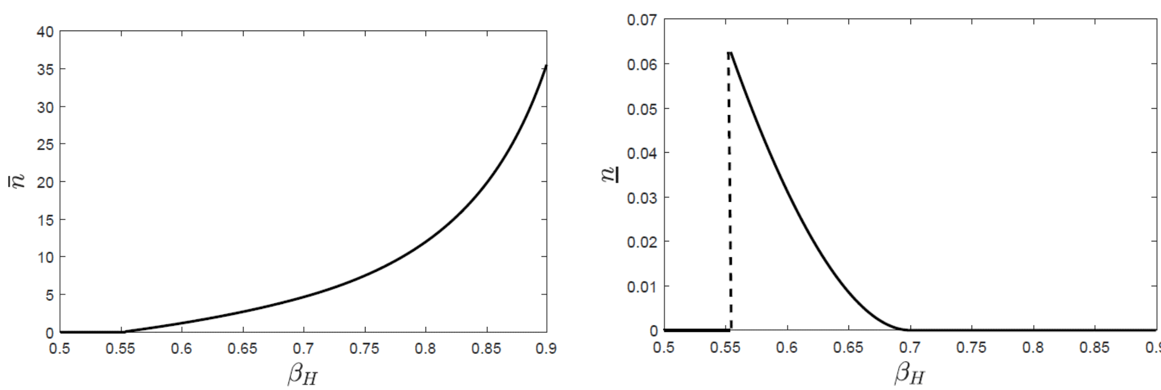
At a high level, this result is intuitive. Games of chance (games with high entropy) leave players with uncertainty about where they will end up after their attempt. Thus, there can be a lot of value for players to wait and see if they can actually make use of bonus actions after their initial attempt fails. Since this uncertainty is resolved when the spot market is reached, it can be difficult for firms to capture value in both the advance and the spot markets in the HAS strategy through differential pricing. In PAS, the spot market is eliminated. With no spot market, the high levels of entropy must be "insured" against ex-ante, allowing for a relatively high advance price selling to a larger proportion of players.

This result has some interesting implications. Consider a game like *Wordscapes* that requires rapidly making words from an arrangement of letters. Although randomness is a factor (the available letters are randomly drawn), there is a high degree of skill involved in the game. This suggests that δ is very low, leading to a narrow range of parameters where the PAS strategy is optimal. This confirms what we see in practice, that *Wordscapes* offers extra time to complete the puzzle throughout the puzzle attempt, not just in advance. Skilled players, *ex-ante* are unlikely to feel the need to purchase the booster, but at the end of their attempt, they can see the direct and clear benefit of purchasing one. Low-skill players predictably "come up short" in many of the puzzles, and so can be enticed to purchase early because they can be convinced that they will use a booster regardless.

At the other end of the spectrum are games with a high degree of randomness, such as mobile game implementations of slots, roulette, etc. In these games, our theory predicts that we might see more PAS strategies implemented in practice. This is intuitive. In games like slots or roulette, so much uncertainty is revealed as the game progresses, many players would want to delay in order to purchase bonus actions until after some of this substantial uncertainty is resolved. However, many players will also realize that bonus actions are worthless if they arrive in a disadvantageous position in the game. What our results suggest is that it is more likely to be optimal in these settings to force the purchase of bonus actions *ex-ante* to increase overall revenue, where more players can be induced to purchase.

8.2. Impact of market characteristics

We explore how market heterogeneity in skill impacts the selling strategy. Recall that $\epsilon = \beta_H - \beta_L$ indicates the skill difference. A large ϵ means the market heterogeneity is high. A small ϵ means the market heterogeneity is low. Figure 3 demonstrates that, for a fixed β_L, the upper threshold $\bar{n}$ (when positive) increases in β_H whereas the lower threshold $\underline{n}$ (when positive) decreases in β_H. We formally establish the result in the following proposition.



(a) upper threshold $\bar{n}$ (b) lower threshold $\underline{n}$

Figure 3 The upper and lower thresholds change with β_H. (Fix $\beta_L = 0.5$, $\delta = 0.1$, $v = 1$ and $P_B = 2$)

Proposition 3 *For a given β_L satisfying $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$, the upper threshold $\bar{n}$ (when positive) increases in β_H while the lower threshold $\underline{n}$ (when positive) decreases in β_H. In other words, as the market heterogeneity ϵ increases, a firm is less likely to adopt the PAS strategy and more likely to adopt the HAS strategy.*

The logic behind this result is simple. As market heterogeneity increases, the value of discriminatory pricing between the advance and spot markets is enhanced, making hybrid pricing—which takes advantage of this type of price discrimination—more attractive.

This result has interesting implications for our game setting. As players become more familiar with a game, the skill difference can change. One possible interpretation is that skill differences widen over time, as the advantage of skilled players is heightened over time with familiarity with the game. This can happen, for instance, if skilled players persist in playing the game over a longer period of time, with lower-skill players being less familiar with the game and its mechanics.

Another possible interpretation is that skill difference narrows over time since intuition and raw ability become less important as low-skill players learn the "tricks of the trade". Our results show that the trend in skill difference naturally leads to a change in the pricing strategy for bonus actions. If the firm notices skill differences increasing with time, they are more likely to favor a hybrid pricing strategy. If the firm notices skill differences narrowing with time, PAS strategies are more likely to be preferred.

8.3. Combining the effects of δ and ϵ

The previous two results have discussed how changes in δ and ϵ impact the choice of selling strategy. This raises a question of the relative impact of δ and ϵ. For example, if we have a large δ, then Proposition 2 suggests the firm is more likely to adopt a PAS strategy, whereas Proposition 3 suggests that a larger ϵ leads a firm to adopt a HAS strategy. So what happens when both δ and ϵ are large?

We examine this question numerically. Consider the instance illustrated in Figure 4, which is representative of all the numerical instances we generated in extensive experiments. Notice that if δ is sufficiently large, the value of β_H (and thus ϵ) is irrelevant. The firm always adopts a PAS strategy. Whereas, for every choice of β_H, there is a cutoff in the value of δ that demarcates a region where PAS is optimal and regular HAS is optimal. This shows that δ is more powerful than ϵ in determining the optimal strategy. What explains this difference?

The intuition is as follows. When δ is large, both high-skill and low-skill players have a significant enough probability for the ending status of the attempt to be so bad that buying bonus actions in the spot market is not warranted. This reduces the value and profit of bonus actions in the spot market. Large values of ϵ favor hybrid strategies because there is scope for price discrimination between the two groups. However, once δ is sufficiently large, there are reduced opportunities to take advantage of this difference because bonus actions are not useful for those who realize a bad ending status in their attempt. Thus, the discrimination benefit of the hybrid strategy is limited. It is, therefore, optimal for the firm to shut down the spot market and focus its attention on advance sales.

8.4. Player welfare

In this section, we examine the total player welfare denoted as $PW = N_H U_H + N_L U_L$, where U_i indicates the utility of a type i player and N_i indicates the number of type i players $(i = H, L)$. Depending on the firm's selling strategy and player behavior, player utilities U_i can be derived following Section 3.2.

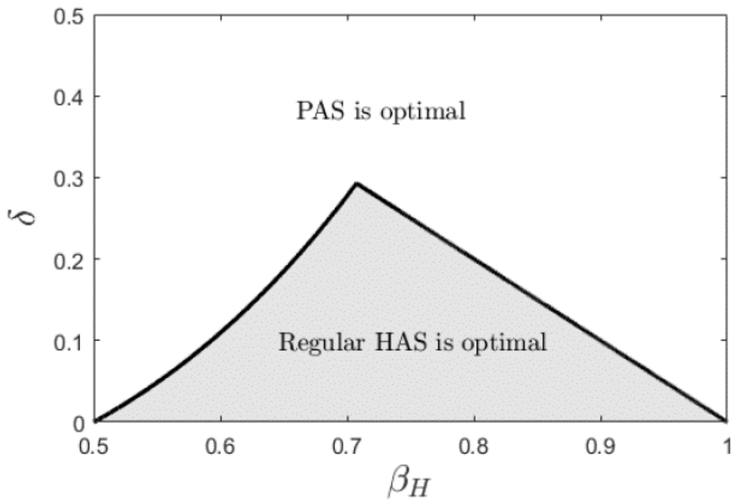


Figure 4 The optimal strategy changes with δ and β_H. (Fix $\beta_L = 0.5$, $v = 1$, $P_B = 2$, and $N_H/N_L = 1$)

Proposition 4 *For casual games, shutting down the spot market is never player welfare maximizing.*

- *The sales strategy that leads to maximum player welfare is PSS.*
- *There exists no "win-win" selling strategy that simultaneously results in the highest profit for the firm and the highest welfare for the players.*

Theorem 1 states that for casual games, the optimal strategy that maximizes the firm's profit would be either the PAS strategy or the regular HAS strategy. Nevertheless, we can show that the PAS strategy, when it is optimal, results in lower player welfare than the regular HAS strategy. We further prove that the PSS strategy leads to a higher player welfare than the regular HAS strategy. This is because having advance sales market open allows the firm to charge a higher spot market price and extract more player surplus. Therefore, there is no win-win strategy for casual games.

This result does, however, shed some possible light onto *Candy Crush*'s choice of only offering bonus actions in a spot market. *Candy Crush* is the flagship game of the developer King, who may be more interested in "growing the base" of people interested in their products than maximizing profit when it comes to their bonus action design. If this is the case, offering bonus actions in the spot market only maximizes player welfare, consistent with a "growth" strategy for the game. Possibly at a later stage of time, King may pursue a more profit-maximizing approach for *Candy Crush* and start to offer bonus actions in the advance sales market.

On the other hand, we find that a win-win scenario can happen for hardcore games. We summarize the finding in the following proposition.

Proposition 5 *For hardcore games, it can be player-welfare maximizing to shut down the spot market.*

- *If the HAS strategies do not exist* [17]*, the PAS strategy is a win-win strategy for the firm and players when the ratio N_H/N_L is moderate.*
- *Otherwise, there is no win-win strategy.*

We have shown in Theorem 2 that for hardcore games, the optimal strategy that maximizes the firm's profit is the PAS strategy. Proposition 5 further indicates that if the HAS strategies do not exist, i.e., neither the regular nor the reverse HAS strategy exists, the pure advance selling strategy leads to the highest player welfare when the ratio N_H/N_L is moderate. When the ratio N_H/N_L is very small or very large, the PSS strategy gives a higher player welfare than the PAS strategy. If the HAS strategy exists (regular hybrid or reverse hybrid), it always results in higher player welfare than the PAS strategy.

Together, Propositions 4 and 5 reveal that it is always player-welfare maximizing to open the spot market in casual games but it may be player-welfare optimal to shut down the spot market in hardcore games. The intuitive reasons for this are straightforward, given the depth of our previous discussions. First, in casual games, bonus actions are valuable to players because β_L and (thus α_L) are likely to be sufficiently high. Thus, when the bonus actions get priced in the spot market, a larger consumer surplus is associated with sales. By contrast, in hardcore games, spot market prices are more likely to target high-skill player valuations and price low-skill players out of the market because their chance of passing the level with bonus actions is low (β_L is low and so α_L is low). Thus, shutting down the spot market and selling at a lower price in the advance market can increase player welfare.

8.5. Alternative pricing mechanisms in the spot market

Our main analysis has proceeded with the assumption that the spot market price p_S can be chosen post-attempt of the player, but that this spot price is uniform across all players. As we have argued, this is consistent with industry practice. Lack of uniformity in pricing is unpopular among players, who often view games as a "level playing field" for progress and, therefore, would find it unfair for some players to receive lower prices than others. The ability to select the spot price post-attempt reflects the game designer's lack of price commitment. They can always reduce the spot price by some discount factor, which is also common in practice. However, the rational expectations assumption invoked in our analysis suggests that players can account for these price adjustments in equilibrium.

In this section, we examine alternative pricing mechanisms in the spot market. First, we examine the impact of personalized pricing in the spot market. Second, we examine what results if the game designer must choose and commit to the spot price before the player attempts the level, at the same time that the advance sales price is determined.

[17] The detailed conditions can be found in Lemma A.4 in the online appendix.

8.5.1. Personalized pricing. First, we assume that the firm can charge a personalized price in the spot market. We get the following result.

Proposition 6 *Suppose the firm charges a personalized price in the spot market.*

- *The optimal spot price will be* $p_S^*(\alpha) = v + \alpha P_B$.
- *The firm achieves the same revenue under the PSS and HAS strategies.*
- *The PSS strategy and the HAS strategy dominate the PAS strategy.*

Proposition 6 says that when the firm can charge a personalized price in the spot market, it is optimal for the firm to adopt the PSS strategy or the HAS strategy. This is intuitive because, under personalized pricing, the PSS and HAS strategies allow for perfect price discrimination in the spot market, since the game designer can observe the ending condition α for every player before selecting the spot price. The PAS strategy is never optimal because it misses out on the opportunity to price discriminate.

Our analysis of the uniform spot pricing case reveals that the PSS strategy is never optimal. In the previous section, we discussed this issue in the paragraph following Proposition 4, where we noted that *Candy Crush* pursues a PSS strategy, and one explanation for this is that *Candy Crush* was designed to maximize player welfare. Proposition 6 provides an alternate explanation: *Candy Crush* is still profit-maximizing but is taking advantage of personalized pricing for price discrimination instead of sticking to a uniform spot price. However, there is no evidence we could find of *Candy Crush* ever customizing the price of bonus actions in the spot market to a particular player. This, does not rule out other video game designers pursuing this type of strategy. We continue to contend that personalized pricing strategies remain highly unpopular among players, so firms take a risk if they pursue a PSS policy with personalized prices.

8.5.2. Price commitment. Now consider the scenario where the firm commits prices p_A and p_S prior to the player's attempt at the level. It is easy to see that under a PAS and PSS strategy, the optimal prices in this scenario are the same as in the previous analysis. For the case of PAS, this is trivial since no p_S is selected (and so the timing of when p_S is chosen is irrelevant). For the PSS strategy, the rational expectations equilibrium assumption makes the two cases equivalent. Differences arise in the hybrid cases.

To see what happens in the hybrid case, we focus on the casual game setting and assume the firm commits to prices such that low-skilled players buy in advance while high-skilled players buy in spot. Accordingly, the firm solves the following optimization problem:

$$\max_{p_A \geq 0, p_S \geq 0} \Pi(p_A, p_S) := p_A N_L + p_S N_H (1 - \beta_H) \mathbb{E}[\mathbb{1}(v + \alpha_H P_B - p_S \geq 0)]$$

$$\text{s.t.} \quad p_A \leq (1 - \beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S)^+]\}$$

$$p_A > (1 - \beta_H)\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S)^+]\}.$$

Via analysis nearly identical in the flavor to our previous arguments (and thus omitted for brevity) we derive a result equivalent to Theorem 1, except where the thresholds $\underline{n}$ and $\overline{n}$ have slightly different expressions. In other words, the optimal strategy is either PAS or HAS, depending on the relative proportion of low-skill and high-skill players. Accordingly, much of our interpretation and discussion applies equally well in the price commitment setting as well as our original setting. In addition, we have the following proposition after comparing these two settings and corresponding results.

Proposition 7 *Prices under committment are higher than those without committment. That is,* $p_S^{*,commit} \geq p_S^{*,dynamic}$ *and* $p_A^{*,commit} \geq p_A^{*,dynamic}$.

The intuition for optimal prices to be higher under price commitment is as follows. Under dynamic pricing, the optimal spot price maximizes the second-period profit only. Under price commitment, the firm decides the spot price to maximize both periods' profits. Because raising the second-period price can help the firm to reduce waiting incentives and thus improve profitability in the first period, the optimal spot price under price commitment is higher than that under dynamic pricing. Correspondingly, the firm can charge a relatively higher advance selling price as well under price commitment.

Proposition 7 indicates that the firm can charge higher prices under committment, implying a higher profit for the regular hybrid strategy. Thus, price committment makes the regular hybrid strategy more likely to be the optimal strategy than in our original setting.[18]

9. Conclusion

In this section, we first summarize the results, followed by the managerial insights obtained in this paper. Then we provide future research directions.

Summary

In this paper, we study how to sell bonus actions in video games. Our results are different for hardcore games and casual games. For hardcore games, the firm should shut down the spot market and adopt the PAS strategy. For casual games, the firm should close, open, and close the spot market (correspondingly, adopt the PAS, hybrid, and PAS strategy) when the market size ratio of high-skill to low-skill players is smaller than, between, and higher than two thresholds, respectively. Furthermore, we find that the two thresholds move towards each other as the game entropy increases

[18] The optimal prices under commitment are characterized in Lemma A.7 of the appendix.

or as the market skill heterogeneity level decreases. Our investigation extends to player welfare and social welfare. We find no win-win strategy exists for casual games, but the PAS strategy can be the win-win strategy for hardcore games.

Managerial Insights

We offered insights in the paragraphs that followed each of our analytical results. However, by assembling and expounding of several of those insights here, we can offer some concrete managerial guidance to game designers. We reserve our insights here for casual games. The case of hardcore games is less nuanced (as illustrated in Theorem 2 and Proposition 5).

Change strategies over the lifecycle of the game: When a game is initially introduced to the market, most players are low skill. So the market size ratio of high skill to low skill is low. But as time goes by, more and more players become high-skill and the market size ratio is more balanced. Eventually, after the game has been released for a long time, most players who stick with the game are "die-hard" fans who tend to be higher skilled. Hence, its market size ratio is high. Therefore, our result suggests that, throughout the life-cycle of a game, the firm should start with a *PAS strategy*, then adopt a *HAS strategy*, finally switching back to a *PAS strategy.*

Evolve strategy as players become more engaged: Usually, a level-based puzzle game starts with easy puzzles to attract low-skill players. However, levels steadily get harder in most games. Thus, as players progress through the levels of the game, higher-skilled players are more rewarded and thus are more likely to stay. Our findings suggest that bonus actions should only be sold in advance at early levels. A HAS strategy is preferred at intermediate levels where low-skill players start to drop-off. Finally, the firm would return to PAS strategy as mostly only experienced players remain and it becomes a hardcore game.

Tune strategy to the randomness of the game design: A game of skill has less randomness (success depends more on skill) than a game of chance. Our findings show that at optimality, the firm is more likely to shut down the spot market and adopt the PAS strategy for a game of chance than for a game of skill.

Adjust strategy if goal is to grow the customer base: From Proposition 4, we learned that profit-maximizing strategies (either HAS or PAS strategies) compromise player welfare. Thus, if the goal of the company is to use bonus actions to grow the customer base (via maximzing player welfare) instead of extracting rents, it is best to pursue a PSS strategy.

Future directions

The model we study can be made more complicated in a number of ways that will bring us even closer to the realism faced by game companies and could be the subjects of future research. For instance, one could expand the model to include player heterogeneity in utilities leading to a more multi-faceted analysis. Other considerations include the possibility of players to *trade* bonus actions among themselves or *gift* them to one another (which is allowable in some games), incorporating a social component into the analysis (see He (2017) for a previous study on trade in video games). There is also the possibility of carrying over unused bonus actions from one level to the next. This consideration would likely demand a dynamic model that incorporates some notion of "inventory". Another future research direction is social comparison. Although players do not interact directly as they attempt the level in a single-player game, they may care about whether they are progressing faster through puzzles than their other friends, or their relative ranking on some leaderboards. This "social comparison" of progress can be an interesting area for future research. Finally, researchers may find interest in unpacking the bundling of bonus actions. For instance, should we sell bonus actions in packages of size three or five? Should we allow for different-sized bundles? All of these questions demonstrate the richness and complexity of the video game setting as a source of opportunities for business researchers.

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Online appendix for "Selling bonus actions in video games"

Appendix A: Derivation of the four selling strategies for casual games

We consider casual games for which β_L is sufficiently high. Specifically, we assume $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$. Below, we characterize the optimal prices and revenue under the four selling strategies (pure advance, pure spot, regular hybrid, and reverse hybrid). The optimal revenues under each selling strategy are denoted as Π^A, Π^S, Π^H, and Π^{RH} respectively. Without causing confusions, we denote the optimal prices as p_A^* and p_S^* without specifying the selling strategies. We let $\epsilon = \beta_H - \beta_L$.

Reverse HAS strategy

As we assume $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$, Lemma 2 implies that a reverse HAS strategy does not exist for casual games. That is, the firm can never set prices p_A and p_S such that low-skill players prefer buying in the spot but high-skill players prefer buying in advance. In this case, we simply let $\Pi^{RH} = 0$. ■

PAS strategy

Lemma A.1 *For casual games, if the firm commits to selling bonus actions only before the attempt, the optimal advance purchase price is*

$$p_A^* = \begin{cases} (1-\beta_L)(\beta_L P_B + v), & \text{if } N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L \\ (1-\beta_H)(\beta_H P_B + v), & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L. \end{cases}$$

The corresponding optimal revenue is

$$\Pi^A = \begin{cases} (1-\beta_L)(\beta_L P_B + v) N_L, & \text{if } N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L \\ (1-\beta_H)(\beta_H P_B + v)(N_H + N_L), & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L. \end{cases}$$

Proof of Lemma A.1: When the firm commits to selling bonus actions only before the attempt, a type i player will purchase bonus actions in the advance sales market if and only if $p_A \leq (1-\beta_i)(\beta_i P_B + v)$. The assumption $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$ results in $(1-\beta_L)(\beta_L P_B + v) \geq (1-\beta_H)(\beta_H P_B + v)$. As a result, the firm's optimization problem is given by

$$\max_{p_A \geq 0} \Pi(p_A) = \begin{cases} p_A(N_H + N_L), & \text{if } p_A \leq (1-\beta_H)(\beta_H P_B + v), \\ p_A N_L, & \text{if } (1-\beta_H)(\beta_H P_B + v) < p_A \leq (1-\beta_L)(\beta_L P_B + v), \\ 0, & \text{if } p_A > (1-\beta_L)(\beta_L P_B + v). \end{cases} \quad \text{(A.1)}$$

The firm's revenue is a piece-wise linear increasing function. Thus, the optimal price p_A^* is either $(1-\beta_L)(\beta_L P_B + v)$ or $(1-\beta_H)(\beta_H P_B + v)$, depending on whichever leads to a higher revenue. Therefore, it suffices to compare the revenues under these two candidate prices. We have

$$\Pi\left((1-\beta_L)(\beta_L P_B + v)\right) = (1-\beta_L)(\beta_L P_B + v) N_L,$$

$$\Pi\left((1-\beta_H)(\beta_H P_B + v)\right) = (1-\beta_H)(\beta_H P_B + v)(N_H + N_L).$$

Their difference is equivalent to

$$\begin{aligned}&\Pi\left((1-\beta_L)(\beta_L P_B + v)\right) - \Pi\left((1-\beta_H)(\beta_H P_B + v)\right)\\ &= (\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]N_L - (1-\beta_H)(v + \beta_H P_B)N_H,\end{aligned}$$

from which we conclude that $\Pi((1-\beta_L)(\beta_L P_B + v)) - \Pi((1-\beta_H)(\beta_H P_B + v)) \geq 0$ if and only if $N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L$.

As a result, the optimal advance purchase price will be

$$p_A^* = \begin{cases}(1-\beta_L)(\beta_L P_B + v), & \text{if } N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L\\ (1-\beta_H)(\beta_H P_B + v), & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L.\end{cases}$$

Following (A.1), the corresponding optimal revenue will be

$$\Pi^A = \begin{cases}(1-\beta_L)(\beta_L P_B + v)N_L, & \text{if } N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L,\\ (1-\beta_H)(\beta_H P_B + v)(N_H + N_L), & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L.\end{cases}$$

■

Regular HAS strategy

Lemma A.2 *For casual games, if the firm adopts the regular HAS strategy (that induces low-skilled players purchase before the attempt but high-skilled players purchase after failing the attempt), the optimal spot price is* $p_S^* = \begin{cases}\frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } v+(\beta_H-3\delta)P_B < 0\\ v+(\beta_H-\delta)P_B, & \text{if } v+(\beta_H-3\delta)P_B \geq 0\end{cases}$*, and the optimal advance purchase price is*

$$p_A^* = \begin{cases}(1-\beta_L)\left(v + \beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}\right), & \text{if } v+(\beta_H-3\delta)P_B < 0,\\ (1-\beta_L)(v+\beta_L P_B), & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta,\\ (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}], & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta.\end{cases}$$

The corresponding optimal revenue is

$$\Pi^H = \begin{cases}(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v+(\beta_H-3\delta)P_B < 0,\\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta,\\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]N_L, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta.\end{cases}$$

Proof of Lemma A.2: We solve the problem backwards. The firm first determines the price p_S to maximize its revenue in the spot market where only high-skill players will make purchases. We denote the firm's spot market revenue as Π_S. The firm's optimization problem in the spot market is given by

$$\max_{p_S \geq 0} \Pi_S(p_S) = p_S N_H (1-\beta_H)\mathbb{E}[\mathbb{1}(v + \alpha_H P_B - p_S \geq 0)]$$

$$= \begin{cases} p_S N_H(1-\beta_H), & \text{if } p_S \le v + (\beta_H - \delta)P_B \\ p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}, & \text{if } v + (\beta_H - \delta)P_B < p_S \le v + (\beta_H + \delta)P_B \\ 0 & \text{if } p_S > v + (\beta_H + \delta)P_B. \end{cases}$$

Clearly, $\Pi_S(p_S)$ is continuous. When $p_S \le v + (\beta_H - \delta)P_B$, $\Pi_S(p_S)$ increases in p_S. When $v + (\beta_H - \delta)P_B < p_S \le v + (\beta_H + \delta)P_B$, $\Pi_S(p_S)$ is a concave quadratic function of p_S. We have

$$\frac{d\Pi_S(p_S)}{dp_S} = \frac{d\left(p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}\right)}{dp_S} = N_H(1-\beta_H)\frac{v + (\beta_H + \delta)P_B - 2p_S}{2\delta P_B}.$$

In particular, at $p_S = v + (\beta_H + \delta)P_B$, we obtain $\frac{d\Pi_S}{dp_S}|_{p_S = v + (\beta_H + \delta)P_B} = -N_H(1-\beta_H)\frac{v + (\beta_H + \delta)P_B}{2\delta P_B} < 0$. At $p_S = v + (\beta_H - \delta)P_B$, we obtain $\frac{d\Pi_S}{dp_S}|_{p_S = v + (\beta_H - \delta)P_B} = -N_H(1-\beta_H)\frac{v + (\beta_H - 3\delta)P_B}{2\delta P_B}$ which can be positive or negative.

If $v + (\beta_H - 3\delta)P_B \ge 0$, implying $\frac{d\Pi_S}{dp_S}|_{p_S = v + (\beta_H - \delta)P_B} \le 0$, we can conclude that $\Pi_S(p_S)$ increases in p_S when $p_S \le v + (\beta_H - \delta)P_B$, and $\Pi_S(p_S)$ decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \le v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be $p_S^* = v + (\beta_H - \delta)P_B$.

If $v + (\beta_H - 3\delta)P_B < 0$, implying $\frac{d\Pi_S}{dp_S}|_{p_S = v + (\beta_H - \delta)P_B} > 0$, we can conclude that $\Pi_S(p_S)$ increases in p_S when $p_S \le v + (\beta_H - \delta)P_B$, and $\Pi_S(p_S)$ first increases and then decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \le v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition $\frac{d\Pi_S(p_S)}{dp_S} = 0$. That is, $p_S^* = \frac{v + (\beta_H + \delta)P_B}{2}$.

Given the optimal spot price p_S^*, the firm determines p_A to maximize its revenue from low-skill players in the advance sales market. We denote the firm's revenue in the advance sales market as Π_A. Thus, the optimization problem is given by

$$\begin{aligned} \max_{p_A \ge 0} \ & \Pi_A(p_A) = p_A N_L \\ \text{s.t.} \quad & p_A \le (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]\} \\ & p_A > (1-\beta_H)\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S^*)^+]\}. \end{aligned}$$

Following Lemma 2, since we assume $\beta_L \ge (1-\beta_H) - \frac{v}{P_B}$, there must exist p_A satisfying $(1-\beta_H)\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S^*)^+] < p_A \le (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]\}$. To maximize its revenue, the firm should set p_A as high as possible. Therefore, the optimal advance purchase price should be $p_A = (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]$. More specifically, given $p_S^* = \begin{cases} \frac{v + (\beta_H + \delta)P_B}{2}, & \text{if } v + (\beta_H - 3\delta)P_B < 0 \\ v + (\beta_H - \delta)P_B, & \text{if } v + (\beta_H - 3\delta)P_B \ge 0 \end{cases}$, we are able to derive

$$\begin{aligned} p_A^* &= (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+] \\ &= \begin{cases} (1-\beta_L)\left(v + \beta_L P_B - \frac{[v + (2\beta_L + \delta - \beta_H)P_B]^2}{16\delta P_B}\right), & \text{if } v + (\beta_H - 3\delta)P_B < 0, \\ (1-\beta_L)(v + \beta_L P_B), & \text{if } v + (\beta_H - 3\delta)P_B \ge 0 \text{ and } \beta_H - \beta_L \ge 2\delta, \\ (1-\beta_L)[v + \beta_L P_B - \frac{(2\delta - \beta_H + \beta_L)^2 P_B}{4\delta}], & \text{if } v + (\beta_H - 3\delta)P_B \ge 0 \text{ and } \beta_H - \beta_L < 2\delta. \end{cases} \end{aligned}$$

Finally, the corresponding optimal revenue is

$$\Pi^H = p_A^* N_L + p_S^* N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B - p_S^* \geq 0)]. \quad \blacksquare$$

PSS strategy

Lemma A.3 *For casual games, if the firm commits to selling bonus actions only after the attempts fails, the optimal spot price is*

$$p_S^* = \begin{cases} v+(\beta_L-\delta)P_B, & \text{if } N_H \leq r_1 N_L \\ \delta P_B \frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}, & \text{if } r_1 N_L < N_H \leq r_2 N_L \\ v+(\beta_H-\delta)P_B, & \text{if } r_2 N_L < N_H < r_3 N_L \\ \frac{(1-\beta_H)N_H[v+(\beta_H+\delta)P_B]+(1-\beta_L)N_L[v+(\beta_L+\delta)P_B]}{2[(1-\beta_H)N_H+(1-\beta_L)N_L]}, & \text{if } N_H \geq r_3 N_L, \end{cases}$$

where the three thresholds r_1, r_2, and r_3 are defined in Table 3. The corresponding optimal revenue is

$$\Pi^S = \begin{cases} [v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)], & \text{if } N_H \leq r_1 N_L \\ \frac{[2(1-\beta_H)\delta N_H P_B+(1-\beta_L)N_L(v+(\beta_L+\delta)P_B)]^2}{8(1-\beta_L)\delta N_L P_B}, & \text{if } r_1 N_L < N_H < r_2 N_L \\ [v+(\beta_H-\delta)P_B]N_H(1-\beta_H), & \text{if } r_2 N_L \leq N_H < r_3 N_L \text{ and } \epsilon \geq 2\delta \\ [v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}] & \text{if } r_2 N_L \leq N_H < r_3 N_L \text{ and } \epsilon < 2\delta, \\ \frac{\{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]+N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]\}^2}{8\delta P_B[N_H(1-\beta_H)+N_L(1-\beta_L)]} & \text{if } N_H \geq r_3 N_L. \end{cases}$$

	Suppose $\epsilon = \beta_H - \beta_L < 2\delta$		
	r_1	r_2	r_3
If $v+(2\beta_H-\beta_L-3\delta)P_B \leq 0$	0	0	0
If $v+(\beta_H-3\delta)P_B \leq 0 < v+(2\beta_H-\beta_L-3\delta)P_B$	0	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{(3\delta-\beta_H)P_B-v}$
If $v+(\beta_L-3\delta)P_B \leq 0 < v+(\beta_H-3\delta)P_B$	0	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$	∞
If $0 < v+(\beta_L-3\delta)P_B$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$	∞
	Suppose $\epsilon = \beta_H - \beta_L \geq 2\delta$		
	r_1	r_2	r_3
If $v+(\beta_L-3\delta)P_B > 0$, and $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$	∞
If $v+(\beta_L-3\delta)P_B > 0$, and $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} > \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$	∞
If $v+(\beta_L-3\delta)P_B \leq 0$,	0	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$	∞

Table 3 **Thresholds r_1, r_2, and r_3 for the determining the optimal spot price for hardcore games**

Proof of Lemma A.3: When the firm commits to selling bonus actions only in the spot market, its optimization problem is given by

$$\max_{p_S \geq 0} \Pi(p_S) = p_S\{N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B - p_S \geq 0)] + N_L(1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B - p_S \geq 0)]\}.$$

Since we assume α_i follows a uniform distribution $U[\beta_i-\delta,\beta_i+\delta]$ for $i=H,L$ and $\beta_L<\beta_H$, the firm's revenue $\Pi(p_S)$ will be a piece-wise continuous function. We consider two scenarios: (I) Suppose $\beta_H-\beta_L<2\delta$, which is equivalent to $\beta_H-\delta<\beta_L+\delta$. Then the support of α_H has overlap with that of α_L; (II) Suppose $\beta_H-\beta_L\geq 2\delta$, which is equivalent to $\beta_H-\delta\geq\beta_L+\delta$. Then, the support of α_H does not overlap with that of α_L.

We start with Scenario (I). We explicitly express the firm's revenue $\Pi(p_S)$ to be

$$\begin{aligned}
\Pi(p_S) &= p_S\{N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B-p_S\geq 0)]+N_L(1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B-p_S\geq 0)]\}\\
&= \begin{cases}
p_S\{N_H(1-\beta_H)+N_L(1-\beta_L)\}, & \text{if } p_S\leq v+(\beta_L-\delta)P_B\\
p_S\{N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(\beta_L+\delta-\frac{p_S-v}{P_B})}{2\delta}\}, & \text{if } v+(\beta_L-\delta)P_B<p_S\leq v+(\beta_H-\delta)P_B\\
p_S\{N_H(1-\beta_H)\frac{(\beta_H+\delta-\frac{p_S-v}{P_B})}{2\delta} & \\
\qquad +N_L(1-\beta_L)\frac{(\beta_L+\delta-\frac{p_S-v}{P_B})}{2\delta}\}, & \text{if } v+(\beta_H-\delta)P_B<p_S\leq v+(\beta_L+\delta)P_B\\
p_S N_H(1-\beta_H)\frac{(\beta_H+\delta-\frac{p_S-v}{P_B})}{2\delta}, & \text{if } v+(\beta_L+\delta)P_B<p_S\leq v+(\beta_H+\delta)P_B\\
0, & \text{if } p_S>v+(\beta_H+\delta)P_B.
\end{cases}
\end{aligned} \tag{A.2}$$

It is straightforward to see that the first piece of $\Pi(p_S)$ when $p_S\leq v+(\beta_L-\delta)P_B$ is a linear increasing function of p_S, while the rest three pieces when $v+(\beta_L-\delta)P_B<p_S\leq v+(\beta_H+\delta)P_B$ are concave quadratic functions of of p_S.

To determine the monotonicity of $\Pi(p_S)$, we would like to investigate its first-order derivative at the kink points. Because $\Pi(p_S)$ may not be smooth, we denote $\frac{d\Pi_S(p_S^0+)}{dp_s}$ as the right derivative when p_S approaches to p_S^0 from the right, and $\frac{d\Pi_S(p_S^0-)}{dp_s}$ as the left derivative when p_S approaches to p_S^0 from the left. We have

$$\begin{aligned}
&\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s}=N_H(1-\beta_H)+N_L(1-\beta_L),\\
&\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}=N_H(1-\beta_H)-N_L(1-\beta_L)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B},\\
&\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}=N_H(1-\beta_H)-N_L(1-\beta_L)\frac{v+(2\beta_H-\beta_L-3\delta)P_B}{2\delta P_B},\\
&\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}=-N_H(1-\beta_H)\frac{v+(\beta_H-3\delta)P_B}{2\delta P_B}-N_L(1-\beta_L)\frac{v+(2\beta_H-\beta_L-3\delta)P_B}{2\delta P_B},\\
&\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}=-N_H(1-\beta_H)\frac{v+(2\beta_L+\delta-\beta_H)P_B}{2\delta P_B}-N_L(1-\beta_L)\frac{v+(\beta_L+\delta)P_B}{2\delta P_B},\\
&\frac{d\Pi_S([v+(\beta_L+\delta)P_B]+)}{dp_s}=-N_H(1-\beta_H)\frac{v+(2\beta_L+\delta-\beta_H)P_B}{2\delta P_B},\\
&\frac{d\Pi_S([v+(\beta_H+\delta)P_B]-)}{dp_s}=-N_H(1-\beta_H)\frac{v+(\beta_H+\delta)P_B}{2\delta P_B}.
\end{aligned}$$

We make several observations. First, $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s}\geq 0$, meaning that $\Pi(p_S)$ increases in p_S when $p_S\leq v+(\beta_L-\delta)P_B$. Second, Scenario (I) assumes $\beta_H-\beta_L<2\delta$, we obtain $(2\beta_L+$

$\delta - \beta_H) = (\beta_L - \delta) + (\beta_L + 2\delta - \beta_H) > 0$. Therefore, $\frac{d\Pi_S([v+(\beta_H+\delta)P_B]-)}{dp_s} < 0$, $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]+)}{dp_s} < 0$, and $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s} < 0$, which implies that $\Pi(p_S)$ decreases in p_S when $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. Furthermore, we have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$.

When $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, there are four possible cases:

(I-a): Suppose $v + (\beta_L - 3\delta)P_B \leq v + (\beta_H - 3\delta)P_B \leq v + (2\beta_H - \beta_L - 3\delta)P_B \leq 0$.

We obtain $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0$, $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} > 0$, and $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} > 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$, it first increases and then decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, and it decreases in p_S when $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, which is equivalent to

$$\frac{d\Pi(p_S)}{dp_S} = N_H(1 - \beta_H)\frac{v + (\beta_H + \delta)P_B - 2p_S}{2\delta P_B} + N_L(1 - \beta_L)\frac{v + (\beta_L + \delta)P_B - 2p_S}{2\delta P_B} = 0.$$

And we solve $p_S^* = \frac{(1-\beta_H)[v+(\beta_H+\delta)P_B]+(1-\beta_L)[v+(\beta_L+\delta)P_B]}{2[(1-\beta_H)N_H+(1-\beta_L)N_L]}$.

(I-b): Suppose $v + (\beta_L - 3\delta)P_B \leq v + (\beta_H - 3\delta)P_B \leq 0 < v + (2\beta_H - \beta_L - 3\delta)P_B$.

We obtain $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0$. However, $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$ and $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$ (satisfying $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$) can be positive or negative.

(I-b-1): Suppose $N_H \geq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{[(3\delta-\beta_H)P_B-v]}$.

We have $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} \geq 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$, it first increases and then decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, and it decreases in p_S when $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be the same as (I-a), that is $p_S^* = \frac{(1-\beta_H)[v+(\beta_H+\delta)P_B]+(1-\beta_L)[v+(\beta_L+\delta)P_B]}{2[(1-\beta_H)N_H+(1-\beta_L)N_L]}$.

(I-b-2): Suppose $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $0 \geq \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_L - \delta)P_B$, it first increases and then decreases in p_S when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$, and it decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$, which is equivalent to

$$\frac{d\Pi(p_S)}{dp_S} = N_H(1 - \beta_H) + N_L(1 - \beta_L)\frac{v + (\beta_L + \delta)P_B - 2p_S}{2\delta P_B} = 0. \tag{A.3}$$

And we solve $p_S^* = \delta P_B \frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$.

(I-b-3): Suppose $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{(3\delta-\beta_H)P_B-v}$.

We have $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > 0 > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$, and it decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be $p_S^* = v + (\beta_H - \delta)P_B$.

(I-c): Suppose $v+(\beta_L-3\delta)P_B \le 0 < v+(\beta_H-3\delta)P_B \le v+(2\beta_H-\beta_L-3\delta)P_B$.

We obtain $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0$ and $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} < 0$. But $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$ can be positive or negative.

(I-c-1): Suppose $N_H \ge N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} \ge 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be $p_S^* = v+(\beta_H-\delta)P_B$.

(I-c-2): Suppose $N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} < 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$, it first increases and then decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition (A.3), that is $p_S^* = \delta P_B \frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$.

(I-d): Suppose $0 < v+(\beta_L-3\delta)P_B \le v+(\beta_H-3\delta)P_B \le v+(2\beta_H-\beta_L-3\delta)P_B$.

We obtain $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} < 0$. However, $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}$ and $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$ (satisfying $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$) can be positive or negative.

(I-d-1): Suppose $N_H \ge N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} \ge 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be $p_S^* = v+(\beta_H-\delta)P_B$.

(I-d-2): Suppose $N_H \le N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B}$.

We have $0 \ge \frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$, and it decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be $p_S^* = v+(\beta_L-\delta)P_B$.

(I-d-3): Suppose $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B} < N_H < \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s} > 0 > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$, it first increases and then decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition (A.3), that is $p_S^* = \delta P_B \frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$.

Finally, in Scenario (I) with $\beta_H-\beta_L < 2\delta$, we define $r_1 = r_2 = r_3 = 0$ if $v+(\beta_L-3\delta)P_B \le v+(\beta_H-3\delta)P_B \le v+(2\beta_H-\beta_L-3\delta)P_B \le 0$. We define $r_1 = 0$, $r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$, and $r_3 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{(3\delta-\beta_H)P_B-v}$ if $v+(\beta_L-3\delta)P_B \le v+(\beta_H-3\delta)P_B \le 0 < v+(2\beta_H-\beta_L-3\delta)P_B$. We define $r_1 = 0$, $r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$, and $r_3 = \infty$ if $v+(\beta_L-3\delta)P_B \le 0 < v+(\beta_H-3\delta)P_B \le$

$v+(2\beta_H-\beta_L-3\delta)P_B$. We define $r_1=\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B}$, $r_2=\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$, and $r_3=\infty$ if $0<v+(\beta_L-3\delta)P_B\le v+(\beta_H-3\delta)P_B\le v+(2\beta_H-\beta_L-3\delta)P_B$.

From the above analysis, we conclude that the optimal spot price p_S^* will be

$$p_S^*=\begin{cases} v+(\beta_L-\delta)P_B, & \text{if } N_H\le r_1N_L\\ \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}+\frac{v+(\beta_L+\delta)P_B}{2}, & \text{if } r_1N_L<N_H\le r_2N_L\\ v+(\beta_H-\delta)P_B, & \text{if } r_2N_L<N_H<r_3N_L\\ \frac{(1-\beta_H)N_H[v+(\beta_H+\delta)P_B]+(1-\beta_L)N_L[v+(\beta_L+\delta)P_B]}{2[(1-\beta_H)N_H+(1-\beta_L)N_L]}, & \text{if } N_H\ge r_3N_L. \end{cases}$$

■

Next, we consider Scenario (II). We explicitly express the firm's revenue $\Pi(p_S)$ to be

$$\begin{aligned}\Pi(p_S)&=p_S\{N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_HP_B-p_S\ge 0)]+N_L(1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_LP_B-p_S\ge 0)]\}\\ &=\begin{cases} p_S\{N_H(1-\beta_H)+N_L(1-\beta_L)\}, & \text{if } p_S\le v+(\beta_L-\delta)P_B\\ p_S\{N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(\beta_L+\delta-\frac{p_S-v}{P_B})}{2\delta}\}, & \text{if } v+(\beta_L-\delta)P_B<p_S\le v+(\beta_L+\delta)P_B\\ p_SN_H(1-\beta_H), & \text{if } v+(\beta_L+\delta)P_B<p_S\le v+(\beta_H-\delta)P_B\\ p_SN_H(1-\beta_H)\frac{(\beta_H+\delta-\frac{p_S-v}{P_B})}{2\delta}, & \text{if } v+(\beta_H-\delta)P_B<p_S\le v+(\beta_H+\delta)P_B\\ 0, & \text{if } p_S>v+(\beta_H+\delta)P_B. \end{cases}\end{aligned}\tag{A.4}$$

We apply a similar analysis as in Scenario (I). Specifically, we examine the first-order derivative at the kink points. We have

$$\begin{aligned}\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s}&=N_H(1-\beta_H)+N_L(1-\beta_L),\\ \frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}&=N_H(1-\beta_H)-N_L(1-\beta_L)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B},\\ \frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}&=N_H(1-\beta_H)-N_L(1-\beta_L)\frac{v+(\beta_L+\delta)P_B}{2\delta P_B},\\ \frac{d\Pi_S([v+(\beta_L+\delta)P_B]+)}{dp_s}&=N_H(1-\beta_H),\\ \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}&=N_H(1-\beta_H),\\ \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}&=-N_H(1-\beta_H)\frac{v+(\beta_H-3\delta)P_B}{2\delta P_B},\\ \frac{d\Pi_S([v+(\beta_H+\delta)P_B]-)}{dp_s}&=-N_H(1-\beta_H)\frac{v+(\beta_H+\delta)P_B}{2\delta P_B}.\end{aligned}$$

We make several observations. First, $\Pi(p_S)$ increases in p_S when $p_S\le v+(\beta_L-\delta)P_B$ and when $v+(\beta_L+\delta)P_Bp_S\le v+(\beta_H-\delta)P_B$. Second, Scenario (II) assumes $\beta_H-\beta_L\ge 2\delta$ and we also assume $\beta_L\ge\delta$, we obtain $\beta_H\ge\beta_L+2\delta\ge 3\delta$. Hence, $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}<0$, implying that $\Pi(p_S)$ decreases in p_S when $v+(\beta_H-\delta)P_B<p_S\le v+(\beta_H+\delta)P_B$. In addition, we have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}>\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$.

When $v+(\beta_L-\delta)P_B<p_S\le v+(\beta_L+\delta)P_B$, there are two possible cases:

(II-a): Suppose $v+(\beta_L-3\delta)P_B \le 0$.

We obtain $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0$. But $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$ can be positive or negative.

(II-a-1): Suppose $N_H \ge N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s} \ge 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be $p_S^* = v+(\beta_H+\delta)P_B$.

(II-a-2): Suppose $N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s} < 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$, it first increases and then decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_L+\delta)P_B$, then it increases in p_S when $v+(\beta_L+\delta)P_B < p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As we can see, $\Pi(p_S)$ has two peaks at $p_S = \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$ and $p_S = v+(\beta_H-\delta)P_B$.

We compare the firm's revenues at these two peaks which are equal to

$$\Pi\left(v+(\beta_H-\delta)P_B\right) = [v+(\beta_H-\delta)P_B]N_H(1-\beta_H),$$

$$\Pi\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) = \frac{[2(1-\beta_H)\delta N_H P_B + (1-\beta_L)N_L(1+(\beta_L+\delta)P_B)]^2}{8(1-\beta_L)\delta N_L P_B}.$$

In particular, we investigate the ratio of the above revenues which can be simplified to be

$$\frac{\Pi\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right)}{\Pi\left(v+(\beta_H-\delta)P_B\right)} \tag{A.5}$$

$$= \frac{\delta P_B}{2[v+(\beta_H-\delta)P_B]}\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) + \frac{[v+(\beta_L+\delta)P_B]}{2[v+(\beta_H-\delta)P_B]} + \frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B[v+(\beta_H-\delta)P_B]}\left(\frac{N_L(1-\beta_L)}{N_H(1-\beta_H)}\right).$$

One can view (A.5) as a function of $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right)$. It can be easily verify that the ratio (A.5) decreases in $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right)$ when $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) \le \frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$. When $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) = \frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$, we have (A.5) $= \frac{v+(\beta_L+\delta)P_B}{v+(\beta_H+\delta)P_B} \le 1$. Therefore, there exists a unique solution $\hat{x}$ satisfying $0 < \hat{x} < \frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$ and

$$\frac{\delta P_B}{2[v+(\beta_H-\delta)P_B]}\hat{x} + \frac{[v+(\beta_L+\delta)P_B]}{2[v+(\beta_H-\delta)P_B]} + \frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B[v+(\beta_H-\delta)P_B]}\frac{1}{\hat{x}} = 1.$$

We solve out

$$\hat{x} = \frac{[v+(2\beta_H-\beta_L-3\delta)P_B] - 2\sqrt{(\beta_H-\beta_L-2\delta)P_B[v+(\beta_H-\delta)P_B]}}{2\delta P_B}.$$

Finally, when $N_H \le N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$, we have $\Pi\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) \ge \Pi\left(v+(\beta_H-\delta)P_B\right)$. As a result, the optimal spot price will be $p_S^* = \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$.

But when $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$, we have $\Pi\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) < \Pi\left(v+(\beta_H-\delta)P_B\right)$. As a result, the optimal spot price will be $p_S^* = v+(\beta_H-\delta)P_B$.

(II-b): Suppose $v + (\beta_L - 3\delta)P_B > 0$.

Then $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}$ and $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$ can be positive or negative.

(II-b-1): Suppose $N_H \geq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > \frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s} \geq 0$. The result will be the same as (II-a-1).

(II-b-2): Suppose $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}\} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta w}$.

We have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0 > \frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$. Similarly as (II-a-2), $\Pi(p_S)$ has two peaks at $p_S = \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$ and $p_S = v + (\beta_H - \delta)P_B$. We need to investigate the ratio of their corresponding revenues (A.5).

Previously, we have shown that (A.5) decreases in $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right)$ when $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) \leq \frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$. Furthermore, when $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) \leq \hat{x}$, we have (A.5) ≥ 1; and when $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) > \hat{x}$, we have (A.5) < 1.

What is left-over is to compare $\hat{x}$ with $\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$. We are able to show that $\hat{x} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$ if and only if $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

(II-b-2.1): Suppose $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

In this case, whenever $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}\} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta w}$, it implies $N_H > N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$. Hence, we always have $\Pi_S\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) < \Pi_S\left(v + (\beta_H - \delta)P_B\right)$. As a result, the optimal spot price should be $p_S^* = v + (\beta_H - \delta)P_B$.

(II-b-2.2): Suppose $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} > \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

In this case, when $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}\} < N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$, we have $\Pi_S\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) \geq \Pi_S\left(v + (\beta_H - \delta)P_B\right)$. The optimal spot price should be $p_S^* = \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$. But when $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta w}$, we have $\Pi_S\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) < \Pi_S\left(v + (\beta_H - \delta)P_B\right)$. The optimal spot price should be $p_S^* = v + (\beta_H - \delta)P_B$.

(II-b-3): Suppose $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $0 \geq \frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > \frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_L - \delta)P_B$, it decreases in p_S when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, then it increases in p_S when $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$, and it decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. Hence, $\Pi(p_S)$ has two peaks at $p_S = v + (\beta_L - \delta)P_B$ and $p_S = v + (\beta_H - \delta)P_B$.

We compare the firm's revenues at these two peaks which are equal to

$$\Pi\left(v + (\beta_L - \delta)P_B\right) = [v + (\beta_L - \delta)P_B]\{N_H(1-\beta_H) + N_L(1-\beta_L)\},$$

$$\Pi\left(v + (\beta_L - \delta)P_B\right) = [v + (\beta_H - \delta)P_B]N_H(1-\beta_H).$$

It is easy to see that $\Pi(v+(\beta_L-\delta)P_B) \geq \Pi(v+(\beta_H-\delta)P_B)$ if and only if $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$. However, we need to compare the two thresholds $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$ and $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$. It turns out that $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$ may be greater or less than $\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

(II-b-3.1) Suppose $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B]} \geq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

In this case, whenever $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$, it implies that $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B}{(\beta_H-\beta_L)P_B}$. Hence, we always have $\Pi(v+(\beta_L-\delta)P_B) \geq \Pi(v+(\beta_H-\delta)P_B)$. As a result, the optimal spot price should be $p_S^* = v+(\beta_L-\delta)P_B$.

(II-b-3.2) Suppose $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} < \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

In this case, when $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B}{(\beta_H-\beta_L)P_B}$, we have $\Pi(v+(\beta_L-\delta)P_B) \geq \Pi(v+(\beta_H-\delta)P_B)$. The optimal spot price should be $p_S^* = v+(\beta_L-\delta)P_B$. But when $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B}{(\beta_H-\beta_L)P_B} < N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$, we have $\Pi(v+(\beta_L-\delta)P_B) < \Pi(v+(\beta_H-\delta)P_B)$. The optimal spot price should be $p_S^* = v+(\beta_H-\delta)P_B$.

Finally, in Scenario (II) with $\beta_H-\beta_L \geq 2\delta$, we define $r_1 = r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$ if $v+(\beta_L-3\delta)P_B \geq 0$ and $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$. We define $r_1 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$ and $r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$ if $v+(\beta_L-3\delta)P_B \geq 0$ and $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} > \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$. We define $r_1 = 0$ and $r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$ if $v+(\beta_L-3\delta)P_B < 0$. We define $r_3 = \infty$.

From the analysis above, we conclude that the optimal spot price p_S^* will be

$$p_S^* = \begin{cases} v+(\beta_L-\delta)P_B, & \text{if } N_H \leq r_1 N_L \\ \delta P_B \frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}, & \text{if } r_1 N_L < N_H < r_2 N_L \\ v+(\beta_H-\delta)P_B, & \text{if } r_2 N_L \leq N_H < r_3 N_L. \end{cases}$$

■

Lastly, the firm's optimal revenue follows from (A.2) and (A.4). ■

Appendix B: Derivation of the four selling strategies for hardcore games

We consider hardcore games for which β_L is relatively low. Specifically, we assume $\beta_L < (1-\beta_H) - \frac{v}{P_B}$. Below, we characterize the optimal prices and revenue under the four selling strategies (pure advance, pure spot, regular hybrid, and reverse hybrid). Without causing confusions, we denote the optimal revenues under each selling strategy as Π^A, Π^S, Π^H, and Π^{RH} respectively. We denote the optimal prices as p_A^* and p_S^* without specifying the selling strategies. Recall that $\epsilon = \beta_H - \beta_L$.

PSS strategy

Notice that in the proof of Lemma A.3, the assumption $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$ does not play a role at all. In other words, whether β_L is greater or less than $(1-\beta_H) - \frac{v}{P_B}$ does not have an impact on a pure spot strategy. Therefore, the optimal spot price and revenue Π^S will be the same as Lemma A.3. ■

Regular HAS strategy

Lemma A.4 *For hardcore games, the optimal regular HAS strategy exists (i.e., there exist p_A and p_S satisfying (5)-(7)) if and only if one of the following conditions holds:*

(1): $v+(\beta_H-3\delta)P_B \geq 0$, $\beta_H-\beta_L \geq 2\delta$, and $(1-\beta_H)[v+(\beta_H-\delta)P_B] < (1-\beta_L)(v+\beta_L P_B)$;

(2): $v+(\beta_H-3\delta)P_B \geq 0$, $\beta_H-\beta_L < 2\delta$, and $(1-\beta_H)[v+(\beta_H-\delta)P_B] < (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]$;

(3): $v+(\beta_H-3\delta)P_B < 0$, and $(1-\beta_H)[v+\beta_H P_B - \frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}] < (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L-\beta_H+\delta)P_B]^2}{16\delta P_B}]$.

Suppose one of the three conditions hold and the optimal regular HAS strategy exists. The optimal spot price is $p_S^* = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } v+(\beta_H-3\delta)P_B < 0 \\ v+(\beta_H-\delta)P_B, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \end{cases}$, and the optimal advance purchase price is

$$p_A^* = \begin{cases} (1-\beta_L)\left(v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}\right), & \text{if } v+(\beta_H-3\delta)P_B < 0, \\ (1-\beta_L)(v+\beta_L P_B), & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta, \\ (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}], & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta. \end{cases}$$

The corresponding optimal revenue is

$$\Pi^H = \begin{cases} (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v+(\beta_H-3\delta)P_B < 0, \\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta, \\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]N_L, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta. \end{cases}$$

Proof of Lemma A.4: The proof for the optimal spot price p_S^* is the same as Lemma A.2. But when $\beta_L < (1-\beta_H) - \frac{v}{P_B}$, there may not exist any p_A satisfying the IC constraints $(1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+] < p_A \leq (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\}$.

Suppose $v+(\beta_H-3\delta)P_B \geq 0$. Then the optimal spot price is $p_S^* = v+(\beta_H-\delta)P_B$. We have

$$(1-\beta_H)\left\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\right\} = (1-\beta_H)[v+(\beta_H-\delta)P_B],$$

$$(1-\beta_L)\left\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\right\} = \begin{cases} (1-\beta_L)(v+\beta_L P_B), & \text{if } \beta_H-\beta_L \geq 2\delta, \\ (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}], & \text{if } \beta_H-\beta_L < 2\delta. \end{cases}$$

Suppose $v+(\beta_H-3\delta)P_B < 0$. Then the optimal spot price is $p_S^* = \frac{v+(\beta_H+\delta)P_B}{2}$. In addition, we have $v+(\beta_L-\delta)P_B \leq v+(\beta_H-\delta)P_B \leq \frac{v+(\beta_H+\delta)P_B}{2} \leq v+(\beta_L+\delta)P_B \leq v+(\beta_H+\delta)P_B$. Thus,

$$(1-\beta_H)\left\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\right\} = (1-\beta_H)[v+\beta_H P_B - \frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}],$$

$$(1-\beta_L)\left\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\right\} = (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}].$$

For the existence of the optimal regular HAS strategy, equivalently the existence of p_A satisfying $(1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+] < p_A \leq (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\}$, we have to require $(1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+] < (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\}$. Specifically, $(1-\beta_H)[v+(\beta_H-\delta)P_B] < (1-\beta_L)(v+\beta_L P_B)$ when $v+(\beta_H-3\delta)P_B \geq 0$ and $\beta_H-\beta_L \geq 2\delta$; or $(1-\beta_H)[v+(\beta_H-\delta)P_B] < (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]$ when $v+(\beta_H-3\delta)P_B \geq 0$ and $\beta_H-\beta_L < 2\delta$; or $(1-\beta_H)[v+\beta_H P_B - \frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}] < (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L-\beta_H+\delta)P_B]^2}{16\delta P_B}]$ when $v+(\beta_H-3\delta)P_B < 0$.

Finally, if there exists a feasible p_A satisfying the IC constraints, then the optimal advance purchase price should be $p_A^* = (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\}$ which is the same as Lemma A.2. And the corresponding optimal revenue will be the same as Lemma A.2 as well. ■

PAS strategy

Lemma A.5 *For hardcore games, if the firm commits to selling bonus actions only before the attempt, the optimal advance purchase price is*

$$p_A^* = \begin{cases} (1-\beta_L)(v+\beta_L P_B), & \text{if } N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B - v]} N_L \\ (1-\beta_H)(v+\beta_H P_B), & \text{if } N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B - v]} N_L. \end{cases}$$

The corresponding optimal revenue is

$$\Pi^A = \begin{cases} (1-\beta_L)(v+\beta_L P_B)(N_H+N_L), & \text{if } N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B - v]} N_L \\ (1-\beta_H)(v+\beta_H P_B)N_H, & \text{if } N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B - v]} N_L. \end{cases}$$

Proof of Lemma A.5:

Recall that when the firm commits to selling bonus actions only before the attempt, a type i player will purchase bonus actions in the advance sales market if and only if $p_A \leq (1-\beta_i)(\beta_i P_B + v)$. For hardcore games, we assume $\beta_L < (1-\beta_H) - \frac{v}{P_B}$, resulting in $(1-\beta_L)(\beta_L P_B + v) < (1-\beta_H)(\beta_H P_B + v)$. As a result, the firm's optimization problem is given by

$$\max_{p_A \geq 0} \Pi(p_A) = \begin{cases} p_A(N_H+N_L), & \text{if } p_A \leq (1-\beta_L)(\beta_L P_B + v), \\ p_A N_H, & \text{if } (1-\beta_L)(\beta_L P_B + v) < p_A \leq (1-\beta_H)(\beta_H P_B + v), \\ 0, & \text{if } p_A > (1-\beta_H)(\beta_H P_B + v). \end{cases} \tag{A.6}$$

As we can see, the optimal price p_A^* is either $(1-\beta_L)(\beta_L P_B + v)$ or $(1-\beta_H)(\beta_H P_B + v)$, depending on whichever leads to a higher revenue. We compare the revenues under these two candidate prices. We have

$$\Pi((1-\beta_L)(\beta_L P_B + v)) = (1-\beta_L)(\beta_L P_B + v)(N_H+N_L),$$
$$\Pi((1-\beta_H)(\beta_H P_B + v)) = (1-\beta_H)(\beta_H P_B + v)N_H.$$

Their difference is equivalent to

$$\Pi((1-\beta_L)(\beta_L P_B + v)) - \Pi((1-\beta_H)(\beta_H P_B + v))$$

$$= (1-\beta_L)(\beta_L P_B + v)N_L - (\beta_H - \beta_L)[(1-\beta_H-\beta_L)P_B - v]N_H.$$

Notice that $\beta_L < (1-\beta_H) - \frac{v}{P_B}$ is equivalent to $(1-\beta_H-\beta_L)P_B - v > 0$. We conclude that $\Pi((1-\beta_L)(\beta_L P_B + v)) - \Pi((1-\beta_H)(\beta_H P_B + v)) \geq 0$ if and only if $N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L$.

As a result, the optimal advance purchase price is

$$p_A^* = \begin{cases} (1-\beta_L)(v+\beta_L P_B), & \text{if } N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L \\ (1-\beta_H)(v+\beta_H P_B), & \text{if } N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L. \end{cases}$$

Following (A.6), the corresponding optimal revenue will be

$$\Pi^A = \begin{cases} (1-\beta_L)(v+\beta_L P_B)(N_H+N_L), & \text{if } N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L \\ (1-\beta_H)(v+\beta_H P_B)N_H, & \text{if } N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L. \end{cases}$$

■

Reverse HAS strategy

Lemma A.6 *For hardcore games, the optimal reverse HAS strategy exists (i.e., there exist* p_A *and* p_A *satisfying (8)-(10)) if and only if (1)* $\beta_L \leq (1-\beta_H) - \frac{v}{P_B}$*; and (2)* $\beta_L < 3\delta - \frac{v}{P_B}$*; and one of the following conditions holds:*

(3.1) $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0$ *and* $(1-\beta_L)\left[v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}$.

Or,

(3.2) $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$ *and* $(1-\beta_L)\left[v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right]$.

Suppose the conditions hold and the optimal reverse HAS strategy exists. The optimal spot price is $p_S^* = \frac{v+(\beta_L+\delta)P_B}{2}$*, and the optimal advance purchase price is*

$$p_A^* = \begin{cases} (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}, & \text{if } v + (2\beta_H-\beta_L-3\delta)P_B \geq 0, \\ (1-\beta_H)[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}], & \text{if } v + (2\beta_H-\beta_L-3\delta)P_B < 0. \end{cases}$$

The corresponding optimal revenue is

$$\Pi^{RH} = \begin{cases} N_H(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v + (2\beta_H-\beta_L-3\delta)P_B \geq 0, \\ N_H(1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v + (2\beta_H-\beta_L-3\delta)P_B < 0. \end{cases}$$

Proof of Lemma A.6:

First of all, Lemma 2 implies that for the existence of the optimal reverse HAS strategy, we must require $v + (\beta_H + \beta_L - 1)P_B < 0$.

Next, we solve the problem backwards. The firm first determines the price p_S to maximize its revenue in the spot market where only low-skill players will make purchases. Recall that we denote the firm's spot market revenue as Π_S. Thus, the firm's optimization problem is given by

$$\max_{p_S \geq 0} \Pi_S(p_S) = p_S N_L (1-\beta_L)\mathbb{E}[\mathbb{1}(v + \alpha_L P_B - p_S \geq 0)]$$

$$= \begin{cases} p_S N_L(1-\beta_L), & \text{if } p_S \le v + (\beta_L - \delta)P_B, \\ p_S N_L(1-\beta_L)\frac{(\beta_L+\delta-\frac{p_S - v}{P_B})}{2\delta}, & \text{if } v+(\beta_L-\delta)P_B < p_S \le v + (\beta_L+\delta)P_B, \\ 0, & \text{if } p_S > v + (\beta_L+\delta)P_B. \end{cases}$$

The analysis for the optimal spot price p_S^* will be the same as Lemma A.2, except that we change the subscript from H to L. We conclude that if $v + (\beta_L - 3\delta)P_B \ge 0$, the optimal spot price is $p_S^* = v + (\beta_L - \delta)P_B$. If $v + (\beta_L - 3\delta)P_B < 0$, the optimal spot price is $p_S^* = \frac{v+(\beta_L+\delta)P_B}{2}$.

Given the optimal spot price p_S^*, the firm determines p_A to maximize its revenue from high-skill players in the advance sales market. Recall that Π_A represents the firm's revenue in the advance sales market. Therefore, the firm's optimization problem is given by

$$\begin{aligned} \max_{p_A \ge 0} \; & \Pi_A(p_A) = p_A N_H \\ \text{s.t.} \quad & p_A > (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]\} \\ & p_A \le (1-\beta_H)\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S^*)^+]\}. \end{aligned}$$

We examine the existence of p_A satisfying $(1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] < p_A \le (1 - \beta_H)\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S^*)^+]$.

Suppose $v + (\beta_L - 3\delta)P_B \ge 0$. Then, the optimal spot price is $p_S^* = v + (\beta_L - \delta)P_B$. We have

$$\begin{aligned} (1-\beta_L)\left\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]\right\} &= (1-\beta_L)[v + (\beta_L - \delta)P_B], \\ (1-\beta_H)\left\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S^*)^+]\right\} &= (1-\beta_H)[v + (\beta_L - \delta)P_B]. \end{aligned}$$

We obtain $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] \ge (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$. Hence, there cannot exist any p_A satisfying the IC constraints $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] < p_A \le (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$. That is, for the existence of the optimal reverse HAS strategy, we must also require $v + (\beta_L - 3\delta)P_B < 0$, equivalently $\beta_L < 3\delta - \frac{v}{P_B}$.

Suppose $v + (\beta_L - 3\delta)P_B < 0$. Then, the optimal spot price is $p_S^* = \frac{v+(\beta_L+\delta)P_B}{2}$. We have

$$(1-\beta_L)\left\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]\right\} = (1-\beta_L)\left[v + \beta_L P_B - \frac{[v + (\beta_L+\delta)P_B]^2}{16\delta P_B}\right],$$

and

$$\begin{aligned} &(1-\beta_H)\left\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S^*)^+]\right\} \\ &= \begin{cases} (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}, & \text{if } \frac{v+(\beta_L+\delta)P_B}{2} \le v + (\beta_H - \delta)P_B, \\ (1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H - \beta_L+\delta)P_B]^2}{16\delta P_B}\right], & \text{if } \frac{v+(\beta_L+\delta)P_B}{2} > v + (\beta_H - \delta)P_B. \end{cases} \end{aligned}$$

For the existence of the optimal reverse HAS strategy, equivalently the existence of p_A satisfying $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] < p_A \le (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$, we have to require $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] < (1-\beta_H)\{v+\beta_H P_B -$

$\mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$. Specifically, $(1-\beta_L)\left[v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}$ when $\frac{v+(\beta_L+\delta)P_B}{2} \leq v+(\beta_H-\delta)P_B$ (which is also equivalent to $v+(2\beta_H-\beta_L-3\delta)P_B \geq 0$); or $(1-\beta_L)\left[v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right]$ when $\frac{v+(\beta_L+\delta)P_B}{2} > v+(\beta_H-\delta)P_B$ (which is also equivalent to $v+(2\beta_H-\beta_L-3\delta)P_B<0$).

Finally, we summarize the conditions needed to ensure the existence of the optimal reverse HAS strategy: (1) $\beta_L \leq (1-\beta_H)-\frac{v}{P_B}$; and (2) $\beta_L < 3\delta - \frac{v}{P_B}$; and (3) $(1-\beta_L)\left[v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}$ when $v+(2\beta_H-\beta_L-3\delta)P_B \geq 0$, or $(1-\beta_L)\left[v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right]$ when $v+(2\beta_H-\beta_L-3\delta)P_B<0$.

If there exists a feasible p_A satisfying the IC constraints, then the optimal spot price must be $p_S^* = \frac{v+(\beta_L+\delta)P_B}{2}$, and the optimal advance purchase price should be $p_A^* = (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$. More specifically, we obtain

$$
\begin{aligned}
p_A^* =& (1-\beta_H)\left\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\right\}\\
=& \begin{cases} (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B \geq 0,\\ (1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right], & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B < 0. \end{cases}
\end{aligned}
$$

The corresponding optimal revenue is equal to

$$
\begin{aligned}
\Pi^{RH} =& p_A^* N_H + p_S^* N_L (1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B - p_S^* \geq 0)]\\
=& \begin{cases} N_H(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B \geq 0,\\ N_H(1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B < 0. \end{cases}
\end{aligned}
$$

■

Appendix C: Technical proofs for the results in the main paper

Proof of Lemma 1

The proof follows the utility functions (U_i^A, U_i^{NA}, and u_i^S) and the IC and IR constraints. ■

Proof of Lemma 2

We consider the difference $U_i^A - U_i^{NA}$ under a HAS strategy that is equal to

$$
\begin{aligned}
U_i^A - U_i^{NA} &= \{\beta_i P_N + (1-\beta_i)(\beta_i P_B + v) - p_A\} - \{\beta_i P_N + (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+]\}\\
&= (1-\beta_i)(\beta_i P_B + v) - (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+] - p_A.
\end{aligned}
$$

We define $\Delta U_i(p_S) = (1-\beta_i)(\beta_i P_B + v) - (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+]$. More specifically,

$$
\Delta U_i(p_S) = \begin{cases} (1-\beta_i)p_S & \text{if } p_S \leq v+(\beta_i-\delta)P_B\\ (1-\beta_i)[v+\beta_i P_B - \frac{(v+(\beta_i+\delta)P_B - p_S)^2}{4\delta P_B}] & \text{if } v+(\beta_i-\delta)P_B < p_S < v+(\beta_i+\delta)P_B\\ (1-\beta_i)(v+\beta_i P_B) & \text{if } v+(\beta_i+\delta)P_B \leq p_S. \end{cases}
$$

Lemma 1 states that a type i will purchase bonus actions in the advance sales market if and only if $p_A \leq \Delta U_i(p_S)$. In the following, we want to prove that $\Delta U_H(p_S) \leq \Delta U_L(p_S)$ for all p_S if and only if $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$.

First of all, suppose $\Delta U_H(p_S) \leq \Delta U_L(p_S)$ for all p_S. Especially when $p_S \geq v + (\beta_H + \delta)P_B \geq v + (\beta_L + \delta)P_B$, we have

$$\Delta U_L(p_S) - \Delta U_H(p_S) = (1-\beta_L)(v + \beta_L P_B) - (1-\beta_H)(v + \beta_H P_B) = (\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]. \tag{A.7}$$

Therefore, $\Delta U_H(p_S) \leq \Delta U_L(p_S)$ implies $[v + (\beta_H + \beta_L - 1)P_B] \geq 0$, which is equivalent to $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$.

Next, suppose $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$. We would like to show $\Delta U_H(p_S) \leq \Delta U_L(p_S)$ for all p_S. Clearly, when $p_S \leq v + (\beta_L - \delta)P_B \leq v + (\beta_H - \delta)P_B$, we obtain $\Delta U_H(p_S) = (1-\beta_H)p_S \leq \Delta U_L(p_S) = (1-\beta_L)p_S$. Besides, when $p_S \geq v + (\beta_H + \delta)P_B \geq v + (\beta_L + \delta)P_B$, given that $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$, we know from (A.7) that $\Delta U_H(p_S) = (1-\beta_H)(v + \beta_H P_B) \leq \Delta U_L(p_S) = (1-\beta_L)(v + \beta_L P_B)$.

The left-over case is when $v + (\beta_L - \delta)P_B < p_S < v + (\beta_H + \delta)P_B$. We examine the difference $\Delta U_L(p_S) - \Delta U_H(p_S)$. It is straightforward to verify that $\Delta U_L(p_S) - \Delta U_H(p_S)$ is a continuous function of p_S. Moreover, its first-order derivative is equal to

$$\frac{d(\Delta U_L(p_S) - \Delta U_H(p_S))}{dp_S}$$

$$= \begin{cases} -(1-\beta_H) + (1-\beta_L)\frac{v+(\beta_L+\delta)P_B - p_S}{2\delta P_B}, & \text{if } v + (\beta_L - \delta)P_B < p_S \leq \min\{v + (\beta_L + \delta)P_B, v + (\beta_H - \delta)P_B\} \\ -(1-\beta_H), & \text{if } v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B \\ (\beta_H - \beta_L)\frac{v + (\beta_H + \beta_L + \delta - 1)P_B - p_S}{2\delta P_B}, & \text{if } v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B \\ -(1-\beta_H)\frac{v + (\beta_H + \delta)P_B - p_S}{2\delta P_B}, & \text{if } \max\{v + (\beta_L + \delta)P_B, v + (\beta_H - \delta)P_B\} < p_S \leq v + (\beta_H + \delta)P_B. \end{cases}$$

Note that when $\beta_H - \beta_L \geq 2\delta$, the case $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$ cannot happen. When $\beta_H - \beta_L < 2\delta$, the case $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$ cannot happen. Thus, the derivative $\frac{d(\Delta U_L(p_S) - \Delta U_H(p_S))}{dp_S}$ has only three pieces as p_S increases from $v + (\beta_L - \delta)P_B$ to $v + (\beta_H + \delta)P_B$.

The derivative $\frac{d(\Delta U_L(p_S) - \Delta U_H(p_S))}{dp_S}$ is continuous in p_S. Moreover, as p_S increases from $v + (\beta_L - \delta)P_B$ to $v + (\beta_H + \delta)P_B$, the derivative $\frac{d(\Delta U_L(p_S) - \Delta U_H(p_S))}{dp_S}$ is first positive and then becomes negative. It means that the difference $\Delta U_L(p_S) - \Delta U_H(p_S)$ first increases in p_S and then decreases in p_S when $v + (\beta_L - \delta)P_B < p_S < v + (\beta_H + \delta)P_B$.

To sum up, we know that the difference $\Delta U_L(p_S) - \Delta U_H(p_S)$ is continuous, first increasing in p_S and then decreasing in p_S. In addition, at $p_S = v + (\beta_L - \delta)P_B$ and $p_S = v + (\beta_H + \delta)P_B$, we have $\Delta U_L(p_S) - \Delta U_H(p_S) \geq 0$. Therefore, we can conclude that $\Delta U_L(p_S) - \Delta U_H(p_S) \geq 0$ whenever $v + (\beta_L - \delta)P_B < p_S < v + (\beta_H + \delta)P_B$.

Above, we have proven that $\Delta U_L(p_S) - \Delta U_H(p_S) \geq 0$ for all p_S, if and only if $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$. Since $U_H^A - U_H^{NA} = \Delta U_H(p_S) - p_A$ and $U_L^A - U_L^{NA} = \Delta U_L(p_S) - p_A$, we finish the proof of Lemma 2.

■

Proof of Corollary 1

Under a PAS strategy, $U_i^A - U_i^{NA} = (1-\beta_i)(\beta_i P_B + v) - p_A$. Corollary 1 comes from Equation (A.7). ■

Proof of Proposition 1

Following Lemma A.2, we have

$$\Pi^H = \begin{cases} (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v+(\beta_H-3\delta)P_B<0, \\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta \\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]N_L, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta. \end{cases}$$

For sake of presentation, we denote the three expressions of Π^H as Π^{H1}, Π^{H2}, and Π^{H3} respectively.

Following Lemma A.3, we have

$$\Pi^S = \begin{cases} [v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)], & \text{if } N_H \leq r_1 N_L \\ \frac{[2(1-\beta_H)\delta N_H P_B + (1-\beta_L)N_L(v+(\beta_L+\delta)P_B)]^2}{8(1-\beta_L)\delta N_L P_B}, & \text{if } r_1 N_L < N_H < r_2 N_L \\ [v+(\beta_H-\delta)P_B]N_H(1-\beta_H), & \text{if } r_2 N_L \leq N_H < r_3 N_L \text{ and } \epsilon \geq 2\delta \\ [v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}] & \text{if } r_2 N_L \leq N_H < r_3 N_L \text{ and } \epsilon < 2\delta, \\ \frac{\{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]+N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]\}^2}{8\delta P_B[N_H(1-\beta_H)+N_L(1-\beta_L)]} & \text{if } N_H \geq r_3 N_L, \end{cases}$$

where the thresholds r_1, r_2, and r_3 are given in Table 3. Notice that Π^S is a piece-wise function with at most four pieces. We denote the four pieces of Π^S to be Π^{S1}, Π^{S2}, Π^{S31} (when $\epsilon \geq 2\delta$) or Π^{S32} (when $\epsilon < 2\delta$), and Π^{S4}. Besides, it is straightforward to verify that Π^S is continuous in N_H.

We would like to prove $\Pi^H \geq \Pi^S$ for all N_H and N_L. To do so, we first make several observations.

(O1) $\Pi^{H2} \geq \Pi^{S1}$ for all N_H and N_L. Because $\Pi^{H2} = (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L$ and $\Pi^{S1} = [v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)]$.

(O2) $\Pi^{H2} \geq \Pi^{S31}$ for all N_H and N_L. Because $\Pi^{H2} = (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L$ and $\Pi^{S31} = [v+(\beta_H-\delta)P_B]N_H(1-\beta_H)$.

(O3) $\Pi^{H1} \geq \Pi^{S31}$ for all N_H and N_L. We have $\Pi^{H1} = (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L$ and $\Pi^{S31} = [v+(\beta_H-\delta)P_B]N_H(1-\beta_H)$. Both can be viewed as linear functions of N_H. Clearly, the intercept of Π^{H1} is higher than that of Π^{S31}. It suffices to prove the slope of Π^{H1} is also higher than that of Π^{S31}. We have

$$(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} - (1-\beta_H)[v+(\beta_H-\delta)P_B] = (1-\beta_H)\frac{[v+(\beta_H-3\delta)P_B]^2}{8\delta P_B} \geq 0.$$

(O4) $\Pi^{H1} \geq \Pi^{S2}$ at $N_H = 0$ if $v+(\beta_L-3\delta)P_B < 0$. We have $\Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$ and $\Pi^{S2}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. Therefore, we obtain

$$\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - [v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$$

$$<\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}-[v+\beta_L P_B-\frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}]=\frac{(3v+3\beta_L P_B-\delta P_B)(v+\beta_L P_B-3\delta P_B)}{16\delta P_B}.$$

If $v+(\beta_L-3\delta)P_B<0$ and we also have $3v+3\beta_L P_B-\delta P_B>0$, we finally obtain that $\Pi^{H1}\geq\Pi^{S2}$ at $N_H=0$.

(O5) $\Pi^{H2}\geq\Pi^{S2}$ at $N_H=0$ if $v+(\beta_L-3\delta)P_B<0$. We have seen that $\Pi^{S2}|_{N_H=0}=N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. And $\Pi^{H2}|_{N_H=0}=N_L(1-\beta_L)(v+\beta_L P_B)$. Then, we have

$$\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}-(v+\beta_L P_B)=\frac{(v+\beta_L P_B)^2-6\delta P_B(v+\beta_L P_B)+(\delta P_B)^2}{8\delta P_B}.$$

Notice that the quadratic function $x^2-6xy+y^2$ is negative when $y\leq x<3y$. Therefore, if $v+(\beta_L-3\delta)P_B<0$, equivalently $v+\beta_L P_B<3\delta P_B$, and we also have $v+\beta_L P_B\geq\delta P_B$, we conclude that $\Pi^{H2}\geq\Pi^{S2}$ at $N_H=0$.

(O6) $\Pi^{H3}\geq\Pi^{S1}$ for all N_H and N_L if $\beta_H-\beta_L<2\delta$. We have $\Pi^{H3}=(1-\beta_H)[v+(\beta_H-\delta)P_B]N_H+(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]N_L$ and $\Pi^{S1}=[v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)]$, both of which are linear functions of N_H. Clearly, Π^{H3} has a higher slope than Π^{S1}. Moreover, the intercept of Π^{H3} satisfies

$$\begin{aligned}N_L(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]&=N_L(1-\beta_L)[v+\beta_L P_B-\delta P_B+\frac{(\beta_H-\beta_L)(4\delta-\beta_H+\beta_L)}{4\delta}]\\&\geq N_L(1-\beta_L)[v+(\beta_L-\delta)P_B],\end{aligned}$$

where the inequality holds since $\beta_H-\beta_L<2\delta<4\delta$. Thus, Π^{H3} also has a higher intercept than Π^{S1}. We conclude that if $\beta_H-\beta_L<2\delta$, $\Pi^{H3}\geq\Pi^{S1}$ for all N_H and N_L.

(O7) $\Pi^{H3}\geq\Pi^{S32}$ for all N_H and N_L. We have $\Pi^{H3}=(1-\beta_H)[v+(\beta_H-\delta)P_B]N_H+(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]N_L$ and $\Pi^{S32}=[v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}]$. Notice that Π^{H3} and Π^{S32}, as functions of N_H, have the same slope. Their intercepts satisfy

$$\begin{aligned}&N_L(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]-N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}]\\=&N_L(1-\beta_L)(\beta_H-\beta_L)\frac{[2v+(\beta_H+\beta_L-2\delta)P_B]}{4\delta}\geq 0,\end{aligned}$$

where the inequality holds because $\beta_H>\beta_L\geq\delta$. Therefore, we conclude that $\Pi^{H3}\geq\Pi^{S32}$ for all N_H and N_L.

(O8) $\Pi^{H3}\geq\Pi^{S2}$ at $N_H=0$ if $v+(\beta_L-3\delta)P_B<0\leq v+(\beta_H-3\delta)P_B$ and $\beta_H-\beta_L<2\delta$. We have $\Pi^{H3}|_{N_H=0}=N_L(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]$. When $v+(\beta_H-3\delta)P_B\geq 0$, we obtain

$$\frac{[v+(2\beta_L+\delta-\beta_H)P_B]}{4\delta}-\frac{(2\delta-\beta_H+\beta_L)P_B}{2\delta}=\frac{v+(\beta_H-3\delta)P_B}{4\delta}\geq 0$$

In addition, when $\beta_H-\beta_L<2\delta$, we have $\frac{[v+(2\beta_L+\delta-\beta_H)P_B]}{4\delta}\geq\frac{(2\delta-\beta_H+\beta_L)P_B}{2\delta}>0$, resulting in

$$\Pi^{H3}|_{N_H=0}=N_L(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]$$

$$>\Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$$
$$>\Pi^{S2}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B},$$

where the last inequality comes from (O4) and we assume $v+(\beta_L-3\delta)P_B<0$.

(O9) Π^{S2} is a convex quadratic function of N_H. And it increases in N_H whenever $N_H \geq 0$. This can be easily seen from the definition of Π^{S2}.

(O10) Π^{S4} is a convex function of N_H. And it increases in N_H whenever $N_H \geq 0$. This is because $\Pi^{S4} = \frac{\{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]+N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]\}^2}{8\delta P_B[N_H(1-\beta_H)+N_L(1-\beta_L)]}$. We are able to show

$$\frac{\partial^2\Pi^{S4}}{\partial N_H^2} = \frac{(1-\beta_H)^2(\beta_H-\beta_L)^2(1-\beta_L)^2N_L^2P_B}{4\delta[(1-\beta_H)N_H+(1-\beta_L)N_L]^3} > 0,$$
$$\frac{\partial\Pi^{S4}}{\partial N_H}|_{N_H=0} = \frac{(1-\beta_H)[v+(\beta_L+\delta)P_B][v+(2\beta_H-\beta_L+\delta)P_B]}{8\delta P_B} > 0,$$
$$\lim_{N_H\to\infty}\frac{\partial\Pi^{S4}}{\partial N_H} = \frac{(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} > 0.$$

(O11) $\Pi^{H1} \geq \Pi^{S4}$ for all N_H and N_L if $v+(\beta_L-3\delta)P_B<0$. Recall that $\Pi^{H1} = (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L$ is a linear function of N_H. Its slope is equal to $(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}$, implying that $\frac{\partial\Pi^{S4}}{\partial N_H} \leq \frac{\partial\Pi^{H1}}{\partial N_H}$ for all N_H. In addition, at $N_H=0$, we have $\Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$ and $\Pi^{S4}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. In (O4), we have already shown that $N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}] \geq N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$ if $v+(\beta_L-3\delta)P_B<0$. From above, we can conclude that $\Pi^{H1} \geq \Pi^{S4}$ for all N_H and N_L when $v+(\beta_L-3\delta)P_B<0$.

(O12) $\Pi^{H1} \geq \Pi^{S32}$ for all N_H and N_L if $v+(\beta_L-3\delta)P_B<0$. Recall that $\Pi^{H1} = (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L$ and $\Pi^{S32} = [v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}]$. Both are linear functions of N_H. We first compare their slopes and we achieve

$$(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} - (1-\beta_H)[v+(\beta_H-\delta)P_B] = (1-\beta_H)\frac{[v+(\beta_H-3\delta)P_B]^2}{8\delta P_B} \geq 0.$$

That is, Π^{H1} has a higher slope than Π^{S32}. Then, we compare their intercepts which are given by $\Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$ and $\Pi^{S32}|_{N_H=0} = N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}$. Notice that

$$\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - [v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta} = \frac{[v+(2\beta_H-\beta_L-3\delta)P_B]^2}{8\delta P_B} \geq 0$$

which implies that

$$\Pi^{S32}|_{N_H=0} = N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}$$

$$\leq \Pi^{S2}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$$
$$\leq \Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}].$$

The last inequality comes from (O4) and we assume $v+(\beta_L-3\delta)P_B<0$. Finally, we can conclude that $\Pi^{H1}\geq\Pi^{S32}$ for all N_H and N_L if $v+(\beta_L-3\delta)P_B<0$.

Given the above observations, we are ready to prove $\Pi^H\geq\Pi^S$ for all N_H and N_L. According to Lemma A.2 and Lemma A.3, we prove the result case by case.

We start with the case with $\epsilon=\beta_H-\beta_L\geq 2\delta$.

(C1.1): If $v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B\geq 0$, then $\Pi^H=\Pi^{H2}$ and $\Pi^S=\begin{cases}\Pi^{S1}, & \text{if } N_H\leq r_1N_L,\\ \Pi^{S2}, & \text{if } r_1N_L<N_H<r_2N_L,\\ \Pi^{S31}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Following (O1) and (O2), we know that $\Pi^H\geq\Pi^S$ when $N_H\leq r_1N_L$ and $N_H\geq r_2N_L$. In particular, $\Pi^H\geq\Pi^S$ at $N_H=r_1N_L$ and $N_H=r_2N_L$. Given that Π^S is continuous and $\Pi^S=\Pi^{S2}$ is convex when $r_1N_L<N_H<r_2N_L$, we further conclude that $\Pi^H\geq\Pi^S$ when $r_1N_L<N_H<r_2N_L$. In summary, we have shown $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon\geq 2\delta$ and $v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B\geq 0$.

(C1.2): If $v+(\beta_H-3\delta)P_B\geq 0>v+(\beta_L-3\delta)P_B$, then $\Pi^H=\Pi^{H2}$ and $\Pi^S=\begin{cases}\Pi^{S2}, & \text{if } 0\leq N_H<r_2N_L,\\ \Pi^{S31}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Similarly as above, (O2) indicates that $\Pi^H\geq\Pi^S$ when $N_H\geq r_2N_L$. In particular, $\Pi^H\geq\Pi^S$ at $N_H=r_2N_L$. Moreover, (O5) indicates that $\Pi^H\geq\Pi^S$ when $N_H=0$, which implies $\Pi^H\geq\Pi^S$ when $0\leq N_H<r_2N_L$. In summary, we have shown $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon\geq 2\delta$ and $v+(\beta_H-3\delta)P_B\geq 0>v+(\beta_L-3\delta)P_B$.

(C1.3): If $0>v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B$, then $\Pi^H=\Pi^{H1}$ and $\Pi^S=\begin{cases}\Pi^{S2}, & \text{if } 0\leq N_H<r_2N_L,\\ \Pi^{S31}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Following (O3) and (O4) and a similar argument as (C1.2), we conclude that $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon\geq 2\delta$ and $0>v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B$.

Above, we have finished the proof for the case with $\epsilon=\beta_H-\beta_L\geq 2\delta$. Next, we consider the case with $\epsilon=\beta_H-\beta_L<2\delta$.

(C2.1): If $v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B\geq 0$, then $\Pi^H=\Pi^{H3}$ and $\Pi^S=\begin{cases}\Pi^{S1}, & \text{if } N_H\leq r_1N_L,\\ \Pi^{S2}, & \text{if } r_1N_L<N_H<r_2N_L,\\ \Pi^{S32}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Following (O6) and (O7) and a similar argument as (C1.1), we conclude that $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon<2\delta$ and $v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B\geq 0$.

(C2.2): If $v+(\beta_H-3\delta)P_B\geq 0>v+(\beta_L-3\delta)P_B$, then $\Pi^H=\Pi^{H2}$ and $\Pi^S=\begin{cases}\Pi^{S2}, & \text{if } 0\leq N_H<r_2N_L,\\ \Pi^{S32}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Following (O7) and (O8) and a similar argument as (C1.2), we conclude that $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon<2\delta$ and $v+(\beta_H-3\delta)P_B\geq 0>v+(\beta_L-3\delta)P_B$.

(C2.3): If $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0 > v + (\beta_H - 3\delta)P_B$, then $\Pi^H = \Pi^{H1}$ and $\Pi^S = \begin{cases} \Pi^{S2}, & \text{if } 0 \leq N_H < r_2 N_L, \\ \Pi^{S32}, & \text{if } r_2 N_L \leq N_H < r_3 N_L \\ \Pi^{S4}, & \text{if } r_3 N_L \leq N_H < \infty. \end{cases}$ Following (O11), (O12), and (O4), we conclude that $\Pi^H \geq \Pi^S$ for all N_H and N_L if $\epsilon < 2\delta$ and $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0 > v + (\beta_H - 3\delta)P_B$.

(C2.4): If $0 > v + (2\beta_H - \beta_L - 3\delta)P_B \geq v + (\beta_H - 3\delta)P_B$, then $\Pi^H = \Pi^{H1}$ and $\Pi^S = \Pi^{S4}$. Following (O11), we conclude that $\Pi^H \geq \Pi^S$ for all N_H and N_L if $\epsilon < 2\delta$ and $0 > v + (2\beta_H - \beta_L - 3\delta)P_B \geq v + (\beta_H - 3\delta)P_B$.

In conclusion, we have discussed all possible cases and shown $\Pi^H > \Pi^S$ for all N_H and N_L. ■

Proof of Theorem 1

For casual games, we assume $\beta_L \geq (1 - \beta_H) - \frac{v}{P_B}$. Lemma 2 implies that the reverse HAS strategy does not exists and Proposition 1 further indicates that the PSS strategy is dominated and can never be optimal. As a result, the optimal selling strategy must be either the PAS strategy or the regular HAS strategy. We compare the firm's revenue under the PAS strategy and the regular HAS strategy which are equal to

$$\Pi^A = \begin{cases} (1-\beta_L)(\beta_L P_B + v)N_L, & \text{if } N_H \leq \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)(v + \beta_H P_B)} N_L \\ (1-\beta_H)(\beta_H P_B + v)(N_H + N_L), & \text{if } N_H > \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)(v + \beta_H P_B)} N_L. \end{cases}$$

$$\Pi^H = \begin{cases} (1-\beta_H)\frac{[v + (\beta_H + \delta)P_B]^2}{8\delta P_B} N_H + (1-\beta_L)[v + \beta_L P_B - \frac{[v + (2\beta_L + \delta - \beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v + (\beta_H - 3\delta)P_B < 0, \\ (1-\beta_H)[v + (\beta_H - \delta)P_B]N_H + (1-\beta_L)(v + \beta_L P_B)N_L, & \text{if } v + (\beta_H - 3\delta)P_B \geq 0, \epsilon \geq 2\delta \\ (1-\beta_H)[v + (\beta_H - \delta)P_B]N_H + (1-\beta_L)[v + \beta_L P_B - \frac{(2\delta - \beta_H + \beta_L)^2 P_B}{4\delta}]N_L, & \text{if } v + (\beta_H - 3\delta)P_B \geq 0, \epsilon < 2\delta. \end{cases}$$

Notice that the firm's revenue under the regular HAS strategy Π^H can be viewed as a linear function of N_H, whereas the firm's the firm's revenue under the PAS strategy Π^A can be viewed as a piece-wise linear function of N_H.

We start with the case that $v + (\beta_H - 3\delta)P_B \geq 0$ and $\beta_H - \beta_L \geq 2\delta$. We consider the difference $\Pi^A - \Pi^H$ which can be simplified to

$$\Pi^A - \Pi^H = \begin{cases} -(1-\beta_H)[v + (\beta_H - \delta)P_B]N_H, & \text{if } N_H \leq \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)(v + \beta_H P_B)} N_L, \\ \delta P_B(1-\beta_H)N_H - (\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]N_L, & \text{if } N_H > \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)(v + \beta_H P_B)} N_L. \end{cases}$$

Clearly, $\Pi^A \leq \Pi^H$ when $N_H \leq \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)(v + \beta_H P_B)} N_L$. When $N_H > \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)(v + \beta_H P_B)} N_L$, we can see from above that $\Pi^A \geq \Pi^H$ if and only if $N_H \geq \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)\delta P_B} N_L$. Lastly, we compare $\frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)\delta P_B}$ with $\frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)(v + \beta_H P_B)}$. Obviously, $\frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)\delta P_B} > \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)(v + \beta_H P_B)}$. Therefore, when $v + (\beta_H - 3\delta)P_B \geq 0$ and $\beta_H - \beta_L \geq 2\delta$, we conclude that $\Pi^A \geq \Pi^H$ if and only if $N_H \geq \frac{(\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]}{(1-\beta_H)\delta P_B} N_L$; otherwise, $\Pi^A < \Pi^H$.

Next, we consider the case that $v + (\beta_H - 3\delta)P_B \geq 0$ and $\beta_H - \beta_L < 2\delta$. Similarly as above, we investigate the difference $\Pi^A - \Pi^H$ which is equal to

$$\Pi^A - \Pi^H$$

$$= \begin{cases} -(1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}N_L, & \text{if } N_H \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L \\ \delta P_B(1-\beta_H)N_H - \{\frac{4\delta(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}\}N_L, & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L. \end{cases}$$

When $N_H \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, we obtain that $\Pi^A \ge \Pi^H$ if and only if $N_H \le \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}N_L$. When $N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, we obtain that $\Pi^A \ge \Pi^H$ if and only if $N_H \ge \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 N_L$.

However, we need to compare $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$ with $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$. We are able to show that $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]} < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$ if and only if $\left(\frac{2\delta-\beta_H+\beta_L}{2\delta}\right)^2 < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)}$. In addition, we need to compare $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ with $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$. We can achieve that $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$ if and only if $\left(\frac{2\delta-\beta_H+\beta_L}{2\delta}\right)^2 < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)}$. Note that $\left(\frac{2\delta-\beta_H+\beta_L}{2\delta}\right)^2$ can be greater or less than $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)}$.

Suppose $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} \le (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$. In this case, we have $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} \le \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, which implies that $\Pi^A \ge \Pi^H$ for all N_H and N_L. Suppose $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} > (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$. In this case, we have $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} > \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$. As a result, from the above discussion, we can conclude that $\Pi^A \ge \Pi^H$ when $N_H \le \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}N_L$ and when $N_H \ge \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 N_L$. And $\Pi^A < \Pi^H$ when $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}N_L < N_H < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 N_L$.

Finally, we examine the case that $v+(\beta_H-3\delta)P_B < 0$. We obtain

$$\Pi^A - \Pi^H$$
$$= \begin{cases} -(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}N_L, & \text{if } N_H \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L, \\ (1-\beta_H)\frac{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + \left\{-(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B] + (1-\beta_L)\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}\right\}N_L, & \\ & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L. \end{cases}$$

When $N_H \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, we obtain that $\Pi^A \ge \Pi^H$ if and only if $N_H \le \frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}N_L$. We have shown earlier in the proof of Proposition 1 that $8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2 > 0$. Thus, when $N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, we obtain that $\Pi^A \ge \Pi^H$ if and only if $N_H \ge \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}N_L$.

Furthermore, we can show that if $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 < \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}$, meaning that $\Pi^H > \Pi^A$ at $N_H = \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, it implies that $\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2} < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} < \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$. As a result, from the above discussion, we conclude that $\Pi^A \ge \Pi^H$ when $N_H \le \frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}N_L$

and when $N_H \geq \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}N_L$. And $\Pi^A < \Pi^H$ when $\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}N_L < N_H < \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}N_L$.

Instead, if $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 \geq \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}$, we have $\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2} \geq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} \geq \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, implying that $\Pi^A \geq \Pi^H$ for all N_H and N_L.

In summary, we define $\underline{n}$ and $\bar{n}$ as follows:

$$\underline{n} = \begin{cases} \frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}, & \text{if } v+(\beta_H-3\delta)P_B<0 \text{ and } \left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 < \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}, \\ \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}, & \text{if } v+(\beta_H-3\delta)P_B\geq 0,\ \beta_H-\beta_L<2\delta, \text{ and} \\ & \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} > (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2, \\ 0, & \text{otherwise.} \end{cases} \tag{A.8}$$

$$\bar{n} = \begin{cases} \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}, & \text{if } v+(\beta_H-3\delta)P_B<0 \text{ and} \\ & \left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 < \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}, \\ \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2, & \text{if } v+(\beta_H-3\delta)P_B\geq 0,\ \beta_H-\beta_L<2\delta, \text{ and} \\ & \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} > (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2, \\ \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}, & \text{if } v+(\beta_H-3\delta)P_B\geq 0 \text{ and } \beta_H-\beta_L\geq 2\delta, \\ 0, & \text{otherwise.} \end{cases} \tag{A.9}$$

We have prove that $\Pi^A \geq \Pi^H$ if and only if $\frac{N_H}{N_L} \leq \underline{n}$ or $\frac{N_H}{N_L} \geq \bar{n}$ while $\Pi^A < \Pi^H$ if and only if $\underline{n} < \frac{N_H}{N_L} < \bar{n}$. ■

Proof of Theorem 2

To prove the theorem, we show the following results:

(1): The PAS strategy dominates the reverse HAS strategy (if exists), i.e., $\Pi^A \geq \Pi^{RH}$.

(2) The PAS strategy dominates the regular HAS strategy (if exists), i.e., $\Pi^A \geq \Pi^H$.

(3) The PAS strategy dominates the pure spot strategy, i.e., $\Pi^A \geq \Pi^S$.

Following Lemma A.5, we have

$$\Pi^A = \begin{cases} (1-\beta_L)(v+\beta_L P_B)(N_H+N_L), & \text{if } N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L \\ (1-\beta_H)(v+\beta_H P_B)N_H, & \text{if } N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L. \end{cases}$$

Notice that Π^A can be viewed as a piece-wise linear function of N_H. For sake of demonstration, we denote the two pieces of Π^A as Π^{A1} and Π^{A2}. Besides, it is straightforward to verify that Π^A is continuous in N_H.

We start with (1) and prove $\Pi^A \geq \Pi^{RH}$ (if Π^{RH} exists and is positive). Following Lemma A.6, we have

$$\Pi^{RH} = \begin{cases} N_H(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B\geq 0, \\ N_H(1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B<0. \end{cases}$$

From the proof of Lemma A.5, we know that $\Pi^A \geq \Pi^{A1} = (1-\beta_L)(v+\beta_L P_B)(N_H+N_L)$ for all N_H and N_L and the strict inequality holds when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L$. Since we want to prove $\Pi^A \geq \Pi^{RH}$ for all N_H and N_L, it suffices to prove $\Pi^{A1} \geq \Pi^{RH}$ for all N_H and N_L.

Suppose $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0$. Then, we have $\Pi^{RH} = N_H(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. The difference $\Pi^{RH} - \Pi^{A1}$ is given by

$$\Pi^{RH} - \Pi^{A1} = N_H\left\{(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} - (1-\beta_L)(v+\beta_L P_B)\right\} + N_L\left\{\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - (1-\beta_L)(v+\beta_L P_B)\right\},$$

which is a linear function of N_L. Its slope satisfies

$$\left\{(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} - (1-\beta_L)(v+\beta_L P_B)\right\} \leq \left\{(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]}{2} - (1-\beta_L)(v+\beta_L P_B)\right\} = -\frac{1}{2}(1-\beta_L)[v+(\beta_L-\delta)P_B] < 0.$$

And its intercept can be simplified to be

$$\left\{\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - (1-\beta_L)(v+\beta_L P_B)\right\} = (1-\beta_L)\frac{(v+\beta_L P_B)^2 - 6\delta P_B(v+\beta_L P_B) + (\delta P_B)^2}{8\delta P_B}.$$

For the existence of the optimal reverse HAS strategy, we need to assume $v + (\beta_L - 3\delta)P_B < 0$. As a result, we have $\delta P_B \leq v + \beta_L P_B < 3\delta P_B$, which implies $(v+\beta_L P_B)^2 - 6\delta P_B(v+\beta_L P_B) + (\delta P_B)^2 < 0$. In conclusion, the slope and the intercept of $\Pi^{RH} - \Pi^{A1}$ are both negative. Hence, when $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0$, we have $\Pi^{RH} < \Pi^{A1} \leq \Pi^A$ for all N_H and N_L.

Similarly as above, when $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$, we have $\Pi^{RH} = N_H(1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. The difference $\Pi^{RH} - \Pi^{A1}$ is equal to

$$\Pi^{RH} - \Pi^{A1} = N_H\left\{(1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] - (1-\beta_L)(v+\beta_L P_B)\right\} + N_L\left\{\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - (1-\beta_L)(v+\beta_L P_B)\right\}.$$

We have already shown that, as a linear function of N_H, the difference $\Pi^{RH} - \Pi^{A1}$ has a negative intercept. Since we assume $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$, we achieve

$$\frac{\partial}{\partial \beta_H}\left\{v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right\} = -\frac{v+(2\beta_H-\beta_L-3\delta)P_B}{4\delta} > 0.$$

Thus, the term $[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}]$ increases in β_H. Given that $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$, equivalently $\beta_H < \frac{(3\delta+\beta_L)P_B - v}{2P_B}$, we obtain

$$\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] \leq \left[v + \left(\frac{(3\delta+\beta_L)P_B - v}{2P_B}\right)P_B - \frac{[v+(2+\left(\frac{(3\delta+\beta_L)P_B-v}{2P_B}\right)-\beta_L+\delta)P_B]^2}{16\delta P_B}\right]$$

$$= \frac{v + (\beta_L + \delta)P_B}{2}.$$

Finally, the slope satisfies

$$\begin{aligned}
&\left\{(1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] - (1-\beta_L)(v+\beta_L P_B)\right\}\\
&\le \left\{(1-\beta_H)\left[\frac{v+(\beta_L+\delta)P_B}{2}\right] - (1-\beta_L)(v+\beta_L P_B)\right\}\\
&\le \left\{(1-\beta_L)\left[\frac{v+(\beta_L+\delta)P_B}{2}\right] - (1-\beta_L)(v+\beta_L P_B)\right\} = -\frac{1}{2}(1-\beta_L)[v+(\beta_L-\delta)P_B] < 0.
\end{aligned}$$

Thus, the slope is also negative. Hence, when $v+(2\beta_H-\beta_L-3\delta)P_B<0$, we conclude $\Pi^{RH}<\Pi^{A1}\le \Pi^A$ for all N_H and N_L.

In summary, we have shown $\Pi^{RH}<\Pi^{A1}\le\Pi^A$ for all N_H and N_L. ■

Next, we prove (2) $\Pi^A \ge \Pi^H$ (if Π^H exists and is positive. Following Lemma A.4, we have

$$\Pi^H = \begin{cases} (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v+(\beta_H-3\delta)P_B<0,\\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L, & \text{if } v+(\beta_H-3\delta)P_B\ge 0,\ \epsilon\ge 2\delta,\\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]N_L, & \text{if } v+(\beta_H-3\delta)P_B\ge 0,\ \epsilon< 2\delta. \end{cases}$$

As in the proof of Proposition 1, we denote the three expressions of Π^H as Π^{H1}, Π^{H2}, and Π^{H3}, all of which are linear functions of N_H.

Clearly, Π^{H1}, Π^{H2}, and Π^{H3} have intercepts no greater than that of Π^{A1}. In addition, as shown in the proof of Proposition 1, we obtain that Π^{H1}, Π^{H2}, and Π^{H3} have smaller slopes than Π^{A2}. Below, we want to show that Π^{H1}, Π^{H2}, and Π^{H3} have smaller slopes than Π^{A1} as well.

According to Lemma A.4, the optimal regular HAS strategy exists if $v+(\beta_H-3\delta)P_B\ge 0$, $\beta_H-\beta_L\ge 2\delta$, and $(1-\beta_H)[v+(\beta_H-\delta)P_B]<(1-\beta_L)(v+\beta_L P_B)$. Therefore, we obtain that Π^{H2} has a smaller slope than Π^{A1}; Or if $v+(\beta_H-3\delta)P_B\ge 0$, $\beta_H-\beta_L\ge 2\delta$, and $(1-\beta_H)[v+(\beta_H-\delta)P_B]<(1-\beta_L)(v+\beta_L P_B)$, from which we know Π^{H3} has a smaller slope than Π^{A1}; Or if $v+(\beta_H-3\delta)P_B<0$, and $(1-\beta_H)[v+\beta_H P_B-\frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}]<(1-\beta_L)[v+\beta_L P_B-\frac{[v+(2\beta_L-\beta_H+\delta)P_B]^2}{16\delta P_B}]$. Given that $v+(\beta_H-3\delta)P_B<0$, we have $v+\beta_H P_B-\frac{3[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}=\frac{-[3(v+\beta_H P_B)\delta P_B][v+\beta_H P_B-3\delta P_B]}{16\delta P_B}>0$. As a result,

$$\begin{aligned}
&(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}\\
\le&(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} + (1-\beta_H)[v+\beta_H P_B - \frac{3[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}] + (1-\beta_L)\frac{[v+(2\beta_L-\beta_H+\delta)P_B]^2}{16\delta P_B}\\
=&(1-\beta_H)[v+\beta_H P_B - \frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}] + (1-\beta_L)\frac{[v+(2\beta_L-\beta_H+\delta)P_B]^2}{16\delta P_B}\\
<&(1-\beta_L)[v+\beta_L P_B).
\end{aligned}$$

We conclude that Π^{H1} has a smaller slope than Π^{A1}.

According to the above discussion, for $j=1,2,3$, we have shown that $\Pi^{A}|_{N_H=0}=\Pi^{A_1}|_{N_H=0}$ is greater than $\Pi^{Hj}|_{N_H=0}$. In addition, Π^{A1} has a higher slope than Π^{Hj}. Therefore, when $N_H \le \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, we always have $\Pi^A=\Pi^{A1}\ge\Pi^{Hj}$. Moreover, Π^{A2} has a higher slope than Π^{Hj}. By continuity, we also know that $\Pi^{A}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} = \Pi^{A_1}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} = \Pi^{A_2}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L}$ is greater than $\Pi^{Hj}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L}$. Thus, when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, we always have $\Pi^A=\Pi^{A2}\ge\Pi^{Hj}$.

In summary, we have shown $\Pi^H\le\Pi^A$ for all N_H and N_L. ■

Finally, we prove (3) $\Pi^A\ge\Pi^S$. Following Lemma A.3, we have

$$\Pi^S=\begin{cases}[v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)], & \text{if } N_H\le r_1N_L\\ \frac{[2(1-\beta_H)\delta N_H P_B+(1-\beta_L)N_L(v+(\beta_L+\delta)P_B)]^2}{8(1-\beta_L)\delta N_L P_B}, & \text{if } r_1N_L<N_H<r_2N_L\\ [v+(\beta_H-\delta)P_B]N_H(1-\beta_H), & \text{if } r_2N_L\le N_H<r_3N_L \text{ and } \epsilon\ge2\delta\\ [v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}] & \text{if } r_2N_L\le N_H<r_3N_L \text{ and } \epsilon<2\delta,\\ \frac{\{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]+N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]\}^2}{8\delta P_B[N_H(1-\beta_H)+N_L(1-\beta_L)]} & \text{if } N_H\ge r_3N_L,\end{cases}$$

where the three thresholds r_1, r_2, and r_3 are defined in Table 3. As before, we denote the four pieces of Π^S as Π^{S1}, Π^{S2}, Π^{S31} (when $\epsilon\ge2\delta$) or Π^{S32} (when $\epsilon<2\delta$), and Π^{S4}.

We make the following observations:

(B1) $\Pi^{A1}\ge\Pi^{S1}$ for all N_H and N_L. Because $\Pi^{A1}=(1-\beta_L)(v+\beta_L P_B)(N_H+N_L)$ and $\Pi^{S1}=[v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)]$.

(B2) $\Pi^{A2}\ge\Pi^{S31}$ for all N_H and N_L. Because $\Pi^{A2}=(1-\beta_H)(v+\beta_H P_B)N_H$ and $\Pi^{S31}=(1-\beta_H)[v+(\beta_H-\delta)P_B]N_H$.

(B3) $\Pi^{A1}\ge\Pi^{S2}$ at $N_H=0$ if $v+(\beta_L-3\delta)P_B<0$. Note that $\Pi^{A1}|_{N_H=0}=N_L(1-\beta_L)(v+\beta_L P_B)$ and $\Pi^{S2}|_{N_H=0}=N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. The proof is the same as (O5) in the proof of Proposition 1.

(B4) $\frac{\partial\Pi^{A2}}{\partial N_H}>\frac{\partial\Pi^{S4}}{\partial N_H}$ for all N_H if $v+(\beta_H-3\delta)P_B<0$. We have shown in (O10) that Π^{S4} is a convex function of N_H. And we are able to show that when $v+(\beta_H-3\delta)P_B<0$,

$$\lim_{N_H\to\infty}\frac{\partial\Pi^{S4}}{\partial N_H}=\frac{(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}<(1-\beta_H)(v+\beta_H P_B)=\frac{\partial\Pi^{A2}}{\partial N_H},$$

which implies $\frac{\partial\Pi^{A2}}{\partial N_H}>\frac{\partial\Pi^{S4}}{\partial N_H}$ for all N_H.

(B5) $\Pi^{A1}\ge\Pi^{S4}$ for all N_H and N_L if $v+(2\beta_H-\beta_L-3\delta)P_B\le0$. First, at $N_H=0$, we have $\Pi^{A1}|_{N_H=0}=(1-\beta_L)(v+\beta_L P_B)N_L$ and $\Pi^{S4}|_{N_H=0}=N_L(1-\beta_L)\frac{[v+(\beta_L+\delta P_B)]^2}{8\delta P_B}$. In the proof of Proposition 1, we have shown $(v+\beta_L P_B)\ge\frac{[v+(\beta_L+\delta P_B)]^2}{8\delta P_B}$. Thus, $\Pi^{A1}|_{N_H=0}\ge\Pi^{S4}|_{N_H=0}$. The condition $v+(2\beta_H-\beta_L-3\delta P_B\le0$ is equivalent to $v+(2\beta_H-\beta_L+\delta P_B\le4\delta P_B$. Therefore,

$$\frac{\partial\Pi^{S4}}{\partial N_H}|_{N_H=0}=\frac{(1-\beta_H)[v+(\beta_L+\delta)P_B][v+(2\beta_H-\beta_L+\delta)P_B]}{8\delta P_B}$$

$$\leq \frac{(1-\beta_H)[v+(\beta_L+\delta)P_B]4\delta P_B}{8\delta P_B} = \frac{(1-\beta_H)[v+(\beta_L+\delta)P_B]}{2} \leq (1-\beta_L)(v+\beta_L P_B).$$

Given that Π^{S4} is convex, we can conclude from the above analysis that $\Pi^{A1} \geq \Pi^{S4}$ for all N_H and N_L if $v+(2\beta_H-\beta_L-3\delta)P_B \leq 0$.

(B6) $\Pi^{A1} \geq \Pi^{S32}$ at $N_H = 0$ and $N_H = \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$. First, $\Pi^{A1}|_{N_H=0} = N_L(1-\beta_L)(v+\beta_L P_B)$ and $\Pi^{S32}|_{N_H=0} = N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta-\beta_H+\beta_L)}{2\delta}$. Recall that we define $\epsilon = \beta_H-\beta_L$. We further have $[v+(\beta_H-\delta)P_B] \geq (\beta_H-\beta_L)P_B = \epsilon P_B$. Finally, we achieve

$$\begin{aligned}
2\delta(v+\beta_L P_B)-[v+(\beta_H-\delta)P_B](2\delta-\beta_H+\beta_L) &= 2\delta[(v+\beta_L P_B)-v-(\beta_H-\delta)P_B]+\epsilon[v+(\beta_H-\delta)P_B] \\
&= 2\delta(\delta-\epsilon)P_B+\epsilon[v+(\beta_H-\delta)P_B] \\
&\geq 2\delta(\delta-\epsilon)P_B+\epsilon^2 P_B = P_B(2\delta^2-2\delta\epsilon+\epsilon^2) \geq 0.
\end{aligned}$$

Equivalently, we have shown $\Pi^{A1}|_{N_H=0} = N_L(1-\beta_L)(v+\beta_L P_B) \geq \Pi^{S32}|_{N_H=0} = N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta-\beta_H+\beta_L)}{2\delta}$.

Next, at $N_H = \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, we have

$$\begin{aligned}
\Pi^{A1}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} &= (1-\beta_H)(v+\beta_H P_B)\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L \\
\Pi^{S32}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} &= (1-\beta_H)[v+(\beta_H-\delta)P_B]\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L \\
&+N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}].
\end{aligned}$$

Their difference is equal to

$$\begin{aligned}
&\Pi^{A1}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} - \Pi^{S32}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} \\
=&(1-\beta_H)\delta P_B\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L-(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}N_L.
\end{aligned}$$

By reorganizing the terms, we can show

$$(1-\beta_H)\delta P_B(v+\beta_L P_B)(2\delta)-[v+(\beta_H-\delta)P_B](2\delta+\beta_L-\beta_H)(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v] \geq 0.$$

As a result, we conclude $\Pi^{A1}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} \geq \Pi^{S32}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L}$. It further implies that $\Pi^{A1} \geq \Pi^{S32}$ when $N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$ because both Π^{A1} and Π^{S32} are linear functions.

Given the above observations, we are able to prove $\Pi^A \geq \Pi^S$ for all N_H and N_L. We prove the result case by case.

We start with the case when $\epsilon = \beta_H-\beta_L \geq 2\delta$. According to Lemma A.3, $\Pi^S = \begin{cases} \Pi^{S1}, & \text{if } N_H \leq r_1 N_L, \\ \Pi^{S2}, & \text{if } r_1 N_L < N_H < r_2 N_L, \\ \Pi^{S31}, & \text{if } r_2 N_L \leq N_H < \infty. \end{cases}$ if $v+(\beta_L-3\delta)P_B \geq 0$. Or $\Pi^S = \begin{cases} \Pi^{S2}, & \text{if } 0 \leq N_H < r_2 N_L, \\ \Pi^{S31}, & \text{if } r_2 N_L \leq N_H < \infty. \end{cases}$ if

$v+(\beta_L-3\delta)P_B<0$. Following (B1), (B2) and (B4), we can conclude that $\Pi^A \geq \Pi^S$ for all N_H and N_L if $\epsilon \geq 2\delta$.

Next, suppose $\epsilon=\beta_H-\beta_L<2\delta$. According to Lemma A.3, if $v+(\beta_L-3\delta)P_B\geq 0$, then $\Pi^S = \begin{cases} \Pi^{S1}, & \text{if } N_H \leq r_1 N_L, \\ \Pi^{S2}, & \text{if } r_1N_L<N_H<r_2N_L, \\ \Pi^{S32}, & \text{if } r_2N_L \leq N_H < \infty. \end{cases}$. Following (B1), we know that $\Pi^A \geq \Pi^S$ when $N_H \leq r_1N_L$. (B6) further implies that $\Pi^{A1}\geq \Pi^{S32}$ when $N_H \leq \frac{(1-\beta_L)(v+\beta_LP_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$ and $\Pi^{A2}\geq\Pi^{S32}$ when $N_H > \frac{(1-\beta_L)(v+\beta_LP_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$. Therefore, $\Pi^A=\max\{\Pi^{A1},\Pi^{A2}\}\geq\Pi^{S32}$ when $r_2N_L\leq N_H<\infty$, which also implies that $\Pi^A\geq\Pi^{S2}$ when $r_1N_L<N_H<r_2N_L$. In conclusion, $\Pi^A\geq\Pi^S$ for all N_H and N_L if $\epsilon<2\delta$ and $v+(\beta_L-3\delta)P_B\geq 0$. Similarly as the proof of Proposition 1, given (B3)-(B6), we are able to show that $\Pi^A\geq\Pi^S$ for all N_H and N_L if $\epsilon<2\delta$ and $v+(\beta_L-3\delta)P_B<0$. Concerning the length of the appendix, we do not repeat the detailed analysis.

In conclusion, from the above analysis, we have shown $\Pi^A\geq\Pi^S$ for all N_H and N_L. ■

Proof of Proposition 2

First of all, we show that $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_HP_B)} - (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ decreases in δ when $2\delta>\beta_H-\beta_L$. It is because its derivative satisfies

$$\frac{\partial}{\partial\delta}\left\{\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_HP_B)}-\left(\frac{2\delta-\beta_H+\beta_L}{2\delta}\right)^2\right\}$$
$$=-\frac{(\beta_H-\beta_L)[2\delta(v+\beta_HP_B)-(\beta_H-\beta_L)(1-\beta_L)P_B]}{2(1-\beta_L)\delta^3P_B}<0,$$

where the last inequality comes from the facts that $2\delta>\beta_H-\beta_L$ and $(v+\beta_HP_B)-(1-\beta_L)P_B=v+(\beta_H+\beta_L-1)P_B\geq v+(2\beta_L-1)P_B>0$. We further obtain that $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 = \left(1-\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right)^2$ increases in δ.

As a result, when δ increases from 0, Theorem 1 indicates that $\underline{n}$ is first zero. When δ is sufficiently large, $\underline{n}$ becomes positive. In particular, $\underline{n}$ is first equal to $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, and when δ is even large, $\underline{n}$ is finally equal to $\left(\frac{1-\beta_L}{2(1-\beta_H)}\right)\left(1-\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right)^2$.

We already have that when $\underline{n}=\left(\frac{1-\beta_L}{2(1-\beta_H)}\right)\left(1-\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right)^2$, it increases in δ. When $\underline{n}=\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, its first-order derivative is given by

$$\frac{\partial\underline{n}}{\partial\delta}=\frac{\partial}{\partial\delta}\left\{\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}\right\}=\frac{P_B(1-\beta_L)(2\delta-\beta_H+\beta_L)[(\beta_H-\beta_L+2\delta)v+(\beta_H^2-\beta_H\beta_L+2\beta_L\delta)P_B]}{4\delta^2(1-\beta_H)[v+(\beta_H-\delta)P_B]^2}.$$

We obtain $\frac{\partial\underline{n}}{\partial\delta}>0$ since $\beta_H>\beta_L$ and $2\delta>\beta_H-\beta_L$.

In summary, we have proven that whenever $\underline{n}$ is positive, it increases in δ. ■

Next, Theorem 1 indicates that as δ increases from 0, $\bar{n}$ is first equal to $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$, then $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$; when δ is quite large, $\bar{n}$ is equal to $\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, and finally $\bar{n}$ becomes zero. Clearly, when $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$, it decreases in δ.

When $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, it suffices to prove $(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ decreasing in δ under the case that $2\delta>\beta_H-\beta_L$. In fact, we have

$$\frac{\partial\left\{(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2\right\}}{\partial\delta}=\frac{(\beta_H-\beta_L)(2\delta-\beta_H+\beta_L)}{2\delta^3}>0.$$

Thus, when $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, it will also decrease in δ.

Finally, when $\bar{n}=\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, its derivative can be simplified to be

$$\begin{aligned}&\frac{\partial\bar{n}}{\partial\delta}\\&=\frac{2P_B(1-\beta_L)[v+(\beta_H+\delta)P_B][v+(2\beta_L+\delta-\beta_H)P_B]}{(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}^2}\times\\&\left\{\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)[v+(\beta_H-\delta)P_B]-(\frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B}-1)(\beta_H-\beta_L)P_B\right\}.\end{aligned}$$

In this case we have $v+(\beta_H-3\delta)P_B<0$ which implies $\beta_H<3\delta<2\beta_L+\delta$. Hence, it suffices to examine the sign of the following term:

$$\left\{\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)[v+(\beta_H-\delta)P_B]-(\frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B}-1)(\beta_H-\beta_L)P_B\right\}. \tag{A.10}$$

Clearly, we have $\frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B}>1$. If $\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)\le 0$, we can easily obtain $\frac{\partial\bar{n}}{\partial\delta}<0$. Suppose $\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)>0$. Following the proof of Theorem 1, $\bar{n}=\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$ if $v+(\beta_H-3\delta)P_B<0$ and $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2<\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}$. As a result, we have

$$\begin{aligned}\frac{\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}}{\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}}&=\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H+\delta)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B](v+\beta_H P_B)}\\&>\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}.\end{aligned}$$

Equivalently,

$$\begin{aligned}&\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)\\&<\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}\right)-\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}\right)\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)\\&=\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}\right)\left(\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right).\end{aligned}$$

Therefore, if $\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B} - \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right) > 0$, we achieve

$$\text{(A.10)} < \left\{\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}\right)\left(\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right)[v+(\beta_H-\delta)P_B] - \left(\frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B}-1\right)(\beta_H-\beta_L)P_B\right\}. \tag{A.11}$$

Since $\delta \le \beta_H < 3\delta$, we further have $\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B} < \frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B} - 1$ and $\frac{2[v+(\beta_H-\delta)P_B]}{v+(\beta_H+\delta)P_B} < 1$, from which we conclude (A.11) < 0, implying that $\frac{\partial \bar{n}}{\partial \delta} < 0$.

In summary, we have proven that $\frac{\partial \bar{n}}{\partial \delta} < 0$ whenever $\bar{n} > 0$, i.e., $\bar{n}$ decreases in δ whenever it is positive. ■

Proof of Proposition 3

First, we argue that when $v + (\beta_H - 3\delta)P_B < 0$, $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 - \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}$ decreases in β_H. Clearly, $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2$ decreases in β_H. In addition, we have

$$\frac{\partial\left\{\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}\right\}}{\partial \beta_H} = \frac{2\left\{[v+(2\beta_H-1)P_B](v+\beta_H P_B) - (\beta_H-\beta_L)P_B[v+(\beta_H+\beta_L-1)P_B]\right\}}{(1-\beta_L)(v+\beta_H P_B)^2} > 0,$$

where the inequality results from the facts that $v + (2\beta_H - 1)P_B \ge (\beta_H - \beta_L)P_B$ and $v + \beta_H P_B \ge v + (\beta_H + \beta_L - 1)P_B$. Next, we argue that whenever $v + (\beta_H - 3\delta)P_B \ge 0$ and $\beta_H - \beta_L < 2\delta$, $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} - (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ increases in β_H. From the above analysis, we can easily obtain $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)}$ increases in β_H, whereas $(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ decreases in β_H when $\beta_H - \beta_L < 2\delta$.

As a result, when β_H increases from $\beta_H = \beta_L$, $\underline{n}$ is first zero. Then it equal to $\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}$. As β_H keeps increasing, $\underline{n}$ becomes $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$. Finally, when β_H is sufficiently large, $\underline{n}$ drops to 0.

When $\underline{n} = \frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}$, we obtain

$$\begin{aligned}\frac{\partial \underline{n}}{\partial \beta_H} = & \frac{(1-\beta_L)[v+(\beta_H+\delta)P_B][v+(2\beta_L+\delta-\beta_H)P_B]}{2\{(1-\beta_H)[v+(\beta_H+\delta)P_B]^2\}^2} \times \\ & \{[v+(2\beta_L+\delta-\beta_H)P_B][v+(3\beta_H+\delta-2)P_B] - 2P_B(1-\beta_H)[v+(\beta_H+\delta)P_B]\}.\end{aligned}$$

The following term determines the sign of $\frac{\partial \underline{n}}{\partial \beta_H}$:

$$[v+(2\beta_L+\delta-\beta_H)P_B][v+(3\beta_H+\delta-2)P_B] - 2P_B(1-\beta_H)[v+(\beta_H+\delta)P_B]. \tag{A.12}$$

Clearly, if $[v + (3\beta_H + \delta - 2)P_B] \le 0$, (A.12) is negative, thus we obtain $\frac{\partial \underline{n}}{\partial \beta_H} \le 0$. In addition, if $[v + (3\beta_H + \delta - 2)P_B] > 0$, we have $v + (2\beta_L + \delta - \beta_H)P_B \le v + (\beta_H + \delta)P_B$ and

$$[v+(3\beta_H+\delta-2)P_B] - 2P_B(1-\beta_H) = v + (\beta_H - 3\delta)P_B + 4(\beta_H+\delta-1)P_B \le 0,$$

where the inequality results from $v+(\beta_H-3\delta)P_B<0$ and $\beta_H+\delta\le 1$. Hence, (A.12), meaning that $\frac{\partial \underline{n}}{\partial \beta_H}\le 0$. In conclusion, whenever $\underline{n}=\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}$, $\underline{n}$ decreases in β_H.

When $\underline{n}=\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, we have

$$\frac{\partial \underline{n}}{\partial \beta_H}=\frac{(1-\beta_L)P_B(2\delta-\beta_H+\beta_L)}{4\delta(1-\beta_H)^2[v+(\beta_H-\delta)P_B]^2}\times\{-2(1-\beta_H)[v+(\beta_H-\delta)P_B]+(2\delta-\beta_H+\beta_L)[v+(2\beta_H-\delta-1)P_B]\}\,.$$

Similarly, it suffices to investigate the sign of the following term:

$$-2(1-\beta_H)[v+(\beta_H-\delta)P_B]+(2\delta-\beta_H+\beta_L)[v+(2\beta_H-\delta-1)P_B]. \tag{A.13}$$

If $[v+(2\beta_H-\delta-1)P_B]\le 0$, (A.13) is negative, thus we obtain $\frac{\partial \underline{n}}{\partial \beta_H}\le 0$. In addition, if $[v+(2\beta_H-\delta-1)P_B]>0$, we have $v+(2\beta_H-\delta-1)P_B\le v+(\beta_H-\delta)P_B$ and $2(1-\beta_H)-(2\delta-\beta_H+\beta_L)=2-\beta_H-\beta_L-2\delta\ge 0$ since $\beta_L+\delta\le\beta_H+\delta\le 1$. Hence, (A.13) is negative, and we achieve $\frac{\partial \underline{n}}{\partial \beta_H}\le 0$. In conclusion, whenever $\underline{n}=\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, $\underline{n}$ decreases in β_H.

In summary, we have proven that whenever $\underline{n}$ is positive. it decreases in β_H. ■

Now, we consider $\bar{n}$. Following Theorem 1, when β_H increases from $\beta_H=\beta_L$, $\bar{n}$ is first zero. Then it equal to $\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$. As β_H keeps increasing, $\bar{n}$ is equal to $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, finally it becomes $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$.

When $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$, it is straightforward to see that $\bar{n}$ increase in β_H. Second, when $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, requiring $\beta_H-\beta_L<2\delta$, we have

$$\frac{\partial\left(\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2\right)}{\partial\beta_H}=-\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)(2-\beta_H-\beta_L-2\delta)}{4(1-\beta_H)^2\delta^2}<0.$$

We conclude that whenever $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, $\bar{n}$ increases in β_H.

Finally, when $\bar{n}=\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, we can easily see that the numerator, $16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2$, increases in β_H. The denominator satisfies

$$\begin{aligned}&\frac{\partial\{(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}{\partial\beta_H}\\&=(\delta P_B)^2-6(\delta P_B)(v+\beta_H P_B)+6(\delta P_B)(1-\beta_H)P_B+(v+\beta_H P_B)^2-2(1-\beta_H)P_B(v+\beta_H P_B),\end{aligned}\tag{A.14}$$

which can be viewed as a convex quadratic function of δP_B. The axis of symmetry is given by $\delta P_B=3[v+\beta_H P_B-(1-\beta_H)P_B]>0$. When $\bar{n}=\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, we must have $\frac{v+\beta_H P_B}{3}<\delta P_B\le v+\beta_H P_B$. We want to show (A.14) is negative whenever $\frac{v+\beta_H P_B}{3}<\delta P_B\le v+\beta_H P_B$. It suffices to check the sign at the two boundary points $\delta P_B=\frac{v+\beta_H P_B}{3}$ and $\delta P_B=v+\beta_H P_B$. In particular, we have

$$\text{(A.14)}|_{\delta P_B=v+\beta_H P_B}=-4(v+\beta_H P_B)[v+(2\beta_H-1)P_B]<0,$$

$$\text{(A.14)}\Big|_{\delta P_B=\frac{v+\beta_H P_B}{3}} = -\frac{8}{9}(v+\beta_H P_B)^2 < 0.$$

where the first inequality results from $v+(2\beta_H-1)P_B \geq v+(\beta_H+\beta_L-1)P_B \geq 0$. Hence, we obtain (A.14) is negative when $\frac{v+\beta_H P_B}{3} < \delta P_B \leq v+\beta_H P_B$, meaning that the denominator of $\bar{n}$ decreases in β_H. Thus, whenever $\bar{n} = \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, $\bar{n}$ increases in β_H.

In summary, we have proven that whenever $\bar{n}$ is positive. it increases in β_H. ■

Proof of Proposition 4

First, we derive the total player welfare under each of the selling strategies for casual games. We define the total player welfare as $PW = N_H U_H + N_L U_L$, where U_H is the utility of a high-type player and U_L is the utility of a low-type player. We use superscript (A, S, and H) to denote the PAS, PSS, and regular HAS strategies respectively.

According to Lemma A.1 and the discussion in Section 3, under the PAS strategy, a type i player receives utilities that follow

$$U_H = \beta_H P_N \quad \text{and} \quad U_L = \begin{cases} \beta_L P_N, & \text{if } p_A^* = (1-\beta_L)(v+\beta_L P_B), \\ \beta_L P_N + (\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B], & \text{if } p_A^* = (1-\beta_H)(v+\beta_H P_B). \end{cases}$$

Under the regular HAS strategy, a type i player receives utilities that follow

$$U_H = \begin{cases} \beta_H P_N + (1-\beta_H)\delta P_B, & \text{if } v+(\beta_H-3\delta)P_B \geq 0, \\ \beta_H P_N + (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}, & \text{if } v+(\beta_H-3\delta)P_B < 0, \end{cases}$$

$$U_L = \begin{cases} \beta_L P_N, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta \\ \beta_L P_N + (1-\beta_L)\frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta \\ \beta_L P_N + (1-\beta_L)\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}, & \text{if } v+(\beta_H-3\delta)P_B < 0. \end{cases}$$

Notice that under the optimal regular HAS strategy, low-type players receive the same utility from purchasing in advance and in the spot market. Thus, the above utility functions can be derived from $\beta_i P_N + (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S^*)^+]$. In other words, under the regular HAS strategy, the utility functions can be expressed as $U_H = \beta_H P_N + (1-\beta_H)\mathbb{E}[(\alpha_H P_B + v - p_S^*)^+]$ and $U_L = \beta_L P_N + (1-\beta_L)\mathbb{E}[(\alpha_L P_B + v - p_S^*)^+]$.

Under the PSS strategy, a type i player receives utilities that should be given by

$$U_H = \beta_H P_N + (1-\beta_H)\mathbb{E}[(\alpha_H P_B + v - p_S^*)^+] \quad \text{and} \quad U_L = \beta_L P_N + (1-\beta_L)\mathbb{E}[(\alpha_L P_B + v - p_S^*)^+]$$

where

$$\mathbb{E}[(\alpha_i P_B + v - p_S^*)^+] = \begin{cases} \beta_i P_B + v - p_S^*, & p_S^* \leq (\beta_i - \delta)P_B + v \\ \frac{[v+(\beta_i+\delta)P_B - p_S]^2}{4\delta P_B}, & (\beta_i-\delta)P_B + v < p_S^* < (\beta_i+\delta)P_B + v \\ 0, & p_S^* \geq (\beta_i+\delta)P_B + v. \end{cases}$$

Following Lemma A.2 and Lemma A.3, we can easily see that the regular HAS strategy charges a higher spot price p_S^* than the PSS strategy. Combining with the above analysis, we conclude that

U_H and U_L will be higher under the PSS strategy than the regular HAS strategy, implying that the total welfare under the regular HAS strategy will be smaller than that under the PSS strategy. Therefore, when $\underline{n}N_L < N_H < \bar{n}N_L$, although the regular HAS strategy maximizes the firm's profit, the total player welfare will be higher under the PSS strategy.

Second, when $N_H \geq \bar{n}N_L$, the total welfare under the PAS strategy is equal to

$$PW^A = \beta_H P_N N_H + \{\beta_L P_N + (\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]\beta_L P_N\} N_L,$$

whereas the total welfare under the HAS strategy is equal to

$$PW^H = \begin{cases} [\beta_H P_N + (1-\beta_H)\delta P_B]N_H + \beta_L P_N N_L, & \text{if } v + (\beta_H - 3\delta)P_B \geq 0,\ \epsilon \geq 2\delta, \\ [\beta_H P_N + (1-\beta_H)\delta P_B]N_H + \{\beta_L P_N + (1-\beta_L)\frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}\}N_L, & \text{if } v + (\beta_H - 3\delta)P_B \geq 0,\ \epsilon < 2\delta, \\ [\beta_H P_N + (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}]N_H + \{\beta_L P_N + (1-\beta_L)\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}\}N_L, & \text{if } v + (\beta_H - 3\delta)P_B < 0. \end{cases}$$

Following the definition of $\bar{n}$, we obtain that $PW^H > PW^A$ when $N_H \geq \bar{n}N_L$. That is, although the PAS strategy results in a higher firm's profit, the regular HAS strategy results in a higher player welfare, further implying from above that the PSS strategy results in the highest player welfare.

Lastly, when $N_H \leq \underline{n}N_L$, the PAS strategy is optimal. The corresponding total player welfare is equal to $PW^A = \beta_H P_N N_H + \beta_L P_N N_L$. Clearly, $PW^A < PW^S$. That is, the PSS strategy results in a higher player welfare than the PAS strategy,

In conclusion, we have proven that the PSS strategy leads to maximal player welfare. Thus, for casual games, there cannot exist a selling strategy that leads to both the highest firm's profit and the highest players' welfare. ■

Proof of Proposition 5

We derive the total player welfare under each of the selling strategies for hardcore games. First, the total welfare under the PSS strategy, denoted as PW^S, is the same as the one in the proof of Proposition 4. In addition, the total welfare under the regular HAS strategy (if exists), denoted as PW^H, is also the same as the one in the proof of Proposition 4.

Following Lemma A.5, under the PAS strategy, a type i player receives utilities that are given by

$$U_H = \begin{cases} \beta_H P_N, & \text{if } p_A^* = (1-\beta_H)(v + \beta_H P_B), \\ \beta_H P_N + (\beta_H - \beta_L)[(1-\beta_H-\beta_L)P_B - v], & \text{if } p_A^* = (1-\beta_L)(v+\beta_L P_B), \end{cases} \quad \text{and} \quad U_L = \beta_L P_N.$$

Under the reverse HAS strategy (if exists), a type i player receives utilities that follow:

$$U_H = \begin{cases} \beta_H P_N + (1-\beta_H)\frac{[v+(2\beta_H-\beta_L-\delta)P_B]}{2}, & \text{if } v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0, \\ \beta_H P_N + (1-\beta_H)\frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}], & \text{if } v + (2\beta_H - \beta_L - 3\delta)P_B < 0. \end{cases}$$

$$U_L = \beta_L P_N + (1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}.$$

We start by showing if the regular or reverse HAS strategy exists, it leads to a higher player welfare than the PAS strategy. That is, $PW^H > PW^A$ and $PW^{RH} > PW^A$. Note that the utility of a low-type player is $U_L = \beta_L P_N$ under the PAS strategy and it is smaller than that under the regular or reverse HAS strategy. In addition, when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, the utility of a high-type player is $U_H = \beta_H P_N$ under the PAS strategy which is also smaller than that under the regular or reverse HAS strategy. Hence, when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, we have $PW^A < PW^H$ and $PW^A < PW^{RH}$.

When $N_H \le \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, the utility of a high-type player is given by $U_H = \beta_H P_N + (1-\beta_H)(v+\beta_H P_B) - p_A^*$. In order to prove $PW^A < PW^H$ and $PW^A < PW^{RH}$, it suffices to prove the advance sale price p_A^* is highest under the PAS strategy. If $N_H \le \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, under the PAS strategy, we have $p_A^* = (1-\beta_L)(v+\beta_L P_B)$. Following Lemma A.4, it is straightforward to see that p_A^* is higher under the PAS strategy than under the regular HAS strategy. Therefore, we conclude $PW^A < PW^H$. Under the reverse HAS strategy, we have $p_A^* = \begin{cases}(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B \ge 0,\\ (1-\beta_H)[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}], & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B < 0.\end{cases}$ Clearly, $(1-\beta_L) \ge (1-\beta_H)$ and $(v+\beta_L P_B) > \frac{[v+(\beta_L+\delta)P_B]}{2}$ since $\beta_L \ge \delta$. Moreover,

$$\frac{\partial[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}]}{\partial \beta_H} = \frac{-[v+(2\beta_H-\beta_L-3\delta)P_B]}{4\delta}.$$

Therefore, if $v+(2\beta_H-\beta_L-3\delta)P_B < 0$, under the reverse HAS strategy, $p_A^* = (1-\beta_H)[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}]$ satisfying

$$\begin{aligned}(1-\beta_H)[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}] &< (1-\beta_H)[(v+\beta_L P_B) - \frac{[v+(2\beta_L-\beta_L+\delta)P_B]^2}{16\delta P_B}]\\ &= (1-\beta_H)[(v+\beta_L P_B) - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}]\\ &< (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}\\ &< (1-\beta_L)(v+\beta_L P_B).\end{aligned}$$

As a result, p_A^* is higher under the PAS strategy than under the reverse HAS strategy. We conclude $PW^{RH} > PW^A$.

So far, we have shown $PW^H > PW^A$ and $PW^{RH} > PW^A$ when the regular or reverse HAS strategy exists. Finally, we prove that when neither the regular or the reverse hybrid exists, under certain conditions, there is a threshold $\bar{t} < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}$ such that $PW^A > PW^S$ if $\bar{t}N_L < N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$.

When $N_H \geq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, following Lemma A.5, we have $p_A^* = (1-\beta_H)(v+\beta_H P_B)$ under the PAS strategy, resulting in

$$PW^A = \beta_H P_N N_H + \beta_L P_N N_L < PW^S.$$

That is, whenever $N_H \geq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, the PSS strategy will lead to a higher player welfare than the PAS strategy.

When $N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, we achieve

$$\begin{aligned} PW^A - PW^S =& (\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B - v]N_H \\ & - (1-\beta_H)\mathbb{E}[(\alpha_H P_B + v - p_S^*)^+]N_H - (1-\beta_L)\mathbb{E}[(\alpha_L P_B + v - p_S^*)^+]N_L, \end{aligned}$$

which can be viewed as a linear function of N_H. Clearly, at $N_H = 0$, $(PW^A - PW^S)|_{N_H=0} < 0$. However, as N_H approaches to $\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, it is possible that $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} > 0$. For example, consider an instance with $P_N = 3$, $P_B = 2$, $\beta_H = 0.3$, $\beta_L = 0.1$ and $\delta = 0.02$. One can verify that $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} > 0$ in this case. For sake of the appendix length, we do not present the algebra.

As a result, if $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} \leq 0$, it implies $PW^A - PW^S < 0$ whenever $N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$. But if $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} > 0$, we can conclude that there exists a threshold $\bar{t}$ such that $PW^A > PW^S$ if $\bar{t}N_L < N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$.

In summary, we have proven that if the optimal HAS strategy exists, it results in a higher player welfare than the PAS strategy. That is, $PW^H > PW^A$ and $PW^{RH} > PW^A$. Therefore, the PAS strategy yields the firm it's highest profit but player welfare is not maximized. If the optimal HAS strategy does not exists, when $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} > 0$, there exists a threshold $\bar{t}$ such that $PW^A > PW^S$ if $\bar{t}N_L < N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$. That is, the PAS strategy is a win-win strategy for the firm and players when the ratio N_H/N_L is moderate. ■

Proof of Proposition 6

Suppose the firm charges a personalized price in the spot market. The PAS strategy shuts down the spot market, thus the optimal PAS strategy is not affected. The optimal revenue under PAS strategy is given in Lemma A.1 for causual games and in Lemma A.5 for hardcare games.

If the firm adopts a PSS strategy, it chooses $p_S(\alpha)$ to maximize its profit from the spot market. Note that the player utility from purchasing bonus actions in the spot market is given by $u^S = v + \alpha P_B - p_S$. As a result, the optimal price that the firm can charge is $p_S^*(\alpha) = v + \alpha P_B$. The corresponding optimal revenue is

$$\Pi^{S,ps} = N_H(1-\beta_H)\mathbb{E}[p_S^*(\alpha_H)] + N_L(1-\beta_L)\mathbb{E}[p_S^*(\alpha_L)] = N_H(1-\beta_H)(v+\beta_H P_B) + N_L(1-\beta_L)(v+\beta_L P_B).$$

If the firm adopts a regular HAS strategy, following the same argument above, the optimal spot price that the firm can charge is $p_S^*(\alpha_H) = v + \alpha_H P_B$. The advance purchase price p_A must satisfy the constraints (6) and (7). Hence, the optimal advance purchase price is $p_A^* = (1-\beta_L)(v+\beta_L P_B)$. The corresponding optimal revenue is

$$\Pi^{H,ps} = N_H(1-\beta_H)\mathbb{E}[p_S^*(\alpha_H)] + N_L p_A^* = N_H(1-\beta_H)(v+\beta_H P_B) + N_L(1-\beta_L)(v+\beta_L P_B).$$

Similarly, if the firm adopts a reverse HAS strategy, the optimal spot price that the firm can charge is $p_S^*(\alpha_L) = v + \alpha_L P_B$. The advance purchase price p_A must satisfy the constraints (9) and (10). Hence, the optimal advance purchase price is $p_A^* = (1-\beta_H)(v+\beta_H P_B)$. The corresponding optimal revenue is

$$\Pi^{RH,ps} = N_H p_A^* + N_L(1-\beta_L)\mathbb{E}[p_S^*(\alpha_L)] = N_H(1-\beta_H)(v+\beta_H P_B) + N_L(1-\beta_L)(v+\beta_L P_B).$$

Finally, for causal games, we assume $\beta_L \geq (1-\beta_H) - \frac{v}{P_B}$. It implies that $(1-\beta_L)(v+\beta_L P_B) \geq (1-\beta_H)(v+\beta_H P_B)$. Therefore, we conclude that $\Pi^{S,ps} = \Pi^{H,ps} \geq \Pi^A$. For hardcore games, we assume $\beta_L < (1-\beta_H) - \frac{v}{P_B}$. It implies that $(1-\beta_L)(v+\beta_L P_B) < (1-\beta_H)(v+\beta_H P_B)$. Therefore, we conclude that $\Pi^{S,ps} = \Pi^{RH,ps} \geq \Pi^A$. ■

Lemma A.7 *For casual games, if the firm commits prices that induces low-skilled players purchase before the attempt but high-skilled players purchase after failing the attempt, the optimal spot price is*

$$p_S^* = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } \epsilon \geq 2\delta \text{ and } v+(\beta_H-3\delta)P_B < 0, \\ \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]}{N_L(1-\beta_L)+2N_H(1-\beta_H)}, & \text{if } \epsilon < 2\delta \text{ and } v+(\beta_H-3\delta)P_B < \frac{N_L(1-\beta_L)}{N_H(1-\beta_H)}(\beta_L+2\delta-\beta_H)P_B, \\ v+(\beta_H-\delta)P_B, & \text{otherwise.} \end{cases}$$

The optimal advance purchase prices satisfies $p_A^* = (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]$.

<u>Proof of Lemma A.7:</u> The firm's optimization problem is given by

$$\max_{p_A \geq 0, p_S \geq 0} \Pi(p_A, p_S) := p_A N_L + p_S N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B - p_S \geq 0)] \tag{A.15}$$

$$\text{s.t.} \quad p_A \leq (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S)^+]\} \tag{A.16}$$

$$p_A > (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S)^+]\}. \tag{A.17}$$

The objective function (A.15) increases in p_A. Hence, p_A must reach the upperbound in (A.16) at optimum. We replace p_A by the upperbound and the objective function becomes a function of p_S that is

$$\text{(A.15)} = N_L(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S)^+]\} + p_S N_H(1-\beta_H)\mathbb{E}[1(v+\alpha_H P_B - p_S \geq 0)].$$

Suppose $\beta_H - \delta \geq \beta_L + \delta$, we have

$$\text{(A.15)} = \begin{cases} N_L(1-\beta_L)p_S + p_S N_H(1-\beta_H), & \text{if } p_S \leq v + (\beta_L - \delta)P_B, \\ N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H), & \text{if } v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H), & \text{if } v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{p_B})}{2\delta}, & \text{if } v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B), & \text{if } p_S > v + (\beta_H + \delta)P_B. \end{cases}$$

We can easily see that (A.15) increases in p_S when $p_S \leq v + (\beta_L - \delta)P_B$ and $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$. Furthermore, we obtain

$$\begin{aligned} &\frac{d}{dp_S}\left\{N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v + (\beta_L + \delta)P_B - p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H)\right\} \\ &= \frac{N_L(1-\beta_L)[v + (\beta_L + \delta)P_B - p_S]}{2\delta P_B} + N_H(1-\beta_H), \end{aligned} \tag{A.18}$$

which is positive when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$. Thus, (A.15) increases in p_S when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$.

Lastly, when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$, we have

$$\frac{d}{dp_S}\left\{N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{p_B})}{2\delta}\right\} = \frac{N_H(1-\beta_H)[v + (\beta_H + \delta)P_B - 2p_S]}{2\delta P_B} \tag{A.19}$$

In particular,

$$\begin{aligned} \text{(A.19)}|_{p_S = v + (\beta_H - \delta)P_B} &= -\frac{N_H(1-\beta_H)[v + (\beta_H - 3\delta)P_B]}{2\delta P_B}, \\ \text{(A.19)}|_{p_S = v + (\beta_H + \delta)P_B} &= -\frac{N_H(1-\beta_H)[v + (\beta_H + \delta)P_B]}{2\delta P_B} < 0. \end{aligned}$$

In conclusion, if $v + (\beta_H - 3\delta)P_B \geq 0$, (A.15) increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$ and decreases in p_S when $p_S > v + (\beta_H - \delta)P_B$. So the optimal spot price is $p_S^* = v + (\beta_H - \delta)P_B$. If $v + (\beta_H - 3\delta)P_B < 0$, (A.15) increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$, increases and then decreases in p_S when $p_S > v + (\beta_H - \delta)P_B$. The optimal spot price is solved from the first-order condition

$$\frac{d}{dp_S}\left\{N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{p_B})}{2\delta}\right\} = \frac{N_H(1-\beta_H)[v + (\beta_H + \delta)P_B - 2p_S]}{2\delta P_B} = 0,$$

which results in $p_S^* = \frac{v + (\beta_H + \delta)P_B}{2}$.

Suppose $\beta_H - \delta < \beta_L + \delta$, we have

$$\text{(A.15)} = \begin{cases} N_L(1-\beta_L)p_S + p_S N_H(1-\beta_H), & \text{if } p_S \leq v + (\beta_L - \delta)P_B, \\ N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H), & \text{if } v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B, \\ N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} \\ \qquad + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{p_B})}{2\delta}, & \text{if } v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{p_B})}{2\delta}, & \text{if } v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B). & \text{if } p_S > v + (\beta_H + \delta)P_B. \end{cases}$$

Clearly, (A.15) increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$. The above analysis further shows that (A.15) increases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_H-\delta)P_B$.

Following the above analysis, when $v+(\beta_L+\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$, the derivative of the objective function is given by (A.19). And we have

$$\text{(A.19)}|_{p_S=v+(\beta_L+\delta)P_B} = -\frac{N_H(1-\beta_H)[v+(2\beta_L+\delta-\beta_H)P_B]}{2\delta P_B} < 0,$$
$$\text{(A.19)}|_{p_S=v+(\beta_H+\delta)P_B} = -\frac{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]}{2\delta P_B} < 0.$$

The first inequality comes from the assumption $\beta_H-\delta < \beta_L+\delta$, equivalently $\beta_H < \beta_L+2\delta < 2\beta_L+\delta$. Therefore, (A.15) decreases in p_S when $v+(\beta_L+\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$.

Finally, when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_L+\delta)P_B$, we have

$$\frac{d}{dp_S}\left\{N_L(1-\beta_L)\{v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B-p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H)\frac{(\beta_H+\delta-\frac{p_S-v}{p_B})}{2\delta}\right\}$$
$$= \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B-p_S]}{2\delta P_B} + \frac{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B-2p_S]}{2\delta P_B}. \tag{A.20}$$

In particular,

$$\text{(A.20)}|_{p_S=v+(\beta_H-\delta)P_B} = \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B}{2\delta P_B} - \frac{N_H(1-\beta_H)[v+(\beta_H-3\delta)P_B]}{2\delta P_B},$$
$$\text{(A.20)}|_{p_S=v+(\beta_L+\delta)P_B} = -\frac{N_H(1-\beta_H)[v+(2\beta_L+\delta-\beta_H)P_B]}{2\delta P_B} < 0.$$

In conclusion, if $N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B \le N_H(1-\beta_H)[v+(\beta_H-3\delta)P_B]$, (A.15) increases in p_S when $p_S \le v+(\beta_H-\delta)P_B$ and decreases in p_S when $p_S > v+(\beta_H-\delta)P_B$. So the optimal spot price is $p_S^* = v+(\beta_H-\delta)P_B$. If $N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B > N_H(1-\beta_H)[v+(\beta_H-3\delta)P_B]$, (A.15) increases in p_S when $p_S \le v+(\beta_H-\delta)P_B$, increases and then decreases in p_S when $p_S > v+(\beta_H-\delta)P_B$. The optimal spot price is solved from the first-order condition

$$0 = \frac{d}{dp_S}\left\{N_L(1-\beta_L)\{v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B-p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H)\frac{(\beta_H+\delta-\frac{p_S-v}{p_B})}{2\delta}\right\}$$
$$= \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B-p_S]}{2\delta P_B} + \frac{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B-2p_S]}{2\delta P_B},$$

which results in $p_S^* = \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}$.

To sum up, following the analysis above, we conclude that the optimal spot price under commitment is

$$p_S^* = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } \epsilon \ge 2\delta \text{ and } v+(\beta_H-3\delta)P_B < 0, \\ \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}, & \text{if } \epsilon < 2\delta \text{ and } v+(\beta_H-3\delta)P_B < \frac{N_L(1-\beta_L)}{N_H(1-\beta_H)}(\beta_L+2\delta-\beta_H)P_B, \\ v+(\beta_H-\delta)P_B, & \text{otherwise.} \end{cases}$$

■

Proof of Proposition 7

Following Lemma A.2 and Lemma A.7, $p_A^{*,dynamic} = (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^{*,dynamic})^+]$ and $p_A^{*,commit} = (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^{*,commit})^+]$. Notice that the function $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S)^+]$ increases in p_S. As a result, it suffices to prove $p_S^{*,commit} \geq p_S^{*,dynamic}$.

We have

$$p_S^{*,dynamic} = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } v+(\beta_H-3\delta)P_B < 0, \\ v+(\beta_H-\delta)P_B, & \text{if } v+(\beta_H-3\delta)P_B \geq 0. \end{cases}$$

and

$$p_S^{*,commit} = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } \epsilon \geq 2\delta \text{ and } v+(\beta_H-3\delta)P_B < 0, \\ \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}, & \text{if } \epsilon < 2\delta \text{ and } v+(\beta_H-3\delta)P_B < \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B}{N_H(1-\beta_H)}, \\ v+(\beta_H-\delta)P_B, & \text{otherwise.} \end{cases}$$

When $\epsilon \geq 2\delta$, we obtain that $p_S^{*,dynamic} = p_S^{*,commit}$ and thereby $p_A^{*,dynamic} = p_A^{*,commit}$.

When $\epsilon < 2\delta$, we have $p_S^{*,dynamic} = p_S^{*,commit} = v + (\beta_H - \delta)P_B$ if $v + (\beta_H - 3\delta)P_B \geq \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B}{N_H(1-\beta_H)}$. If $0 \leq v+(\beta_H-3\delta)P_B < \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B}{N_H(1-\beta_H)}$, $p_S^{*,dynamic} = v+(\beta_H-\delta)P_B$ and $p_S^{*,commit} = \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}$ from which we obtain

$$p_S^{*,commit} - p_S^{*,dynamic} = \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B - N_H(1-\beta_H)[v+(\beta_H-3\delta)P_B]}{N_L(1-\beta_L)+2N_H(1-\beta_H)} > 0.$$

Lastly, if $v + (\beta_H - 3\delta)P_B < 0$, $p_S^{*,dynamic} = \frac{v+(\beta_H+\delta)P_B}{2}$ and $p_S^{*,commit} = \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}$. We assume that $\epsilon = \beta_H - \beta_L < 2\delta$ which implies $\beta_H < \beta_L + 2\delta \leq 2\beta_L + \delta$. Thus,

$$p_S^{*,commit} - p_S^{*,dynamic} = \frac{N_L(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}{2[N_L(1-\beta_L)+2N_H(1-\beta_H)]} > 0.$$

In conclusion, we have shown $p_S^{*,commit} \geq p_S^{*,dynamic}$ and correspondingly $p_A^{*,commit} \geq p_A^{*,dynamic}$. ■