Optimal sequencing in single-player games

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An important problem in single-player video game design is how to sequence game elements within a level (or "chunk") of the game. Each element has two critical features: a *reward* (e.g., earning an item or being able to watch a cinematic) and a degree of *difficulty* (e.g., how much energy or focus is needed to interact with the game element). The latter property is a distinctive feature in video games. Unlike passive services (like a trip to the spa) or passive entertainment (like watching sports or movies), video games often require concerted effort to consume. We study how to sequence game elements to maximize overall experienced utility subject to the dynamics of adaptation to rewards and difficulty as well as memory decay.

We find that the optimal design depends on the relationship between rewards and difficulty, leading to qualitatively different designs. For example, when the proportion of reward-to-difficulty is high, the optimal design mimics that of more passive experiences (as studied in Das Gupta et al. (2016)). By contrast, the optimal design of games with low reward-to-difficulty ratios resembles work-out routines with "warm-ups" and "cool-downs". Intermediate cases may follow the classical "mini-boss, end-boss" design where difficulty has two peaks. Numerical results reveal optimal designs with "waves" of reward and difficulty with multiple peaks. Level designs with multiple peaks of difficulty are ubiquitous in video games. In summary, this paper provides practical guidance to game designers on how to match the design of single-player games to the relationship between reward and difficulty inherent in their game's mechanics. Our model also has implications for other interactive services that share similarities with games, such as summer camps for children.

Key words: video games; level design; memory decay; adaptation

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1. Introduction

Video games are big business, representing the largest and fastest-growing segment of the entertainment industry.[1] However, not all games are successful. One example of a failed game — *E.T.*

[1] https://www.reuters.com/sponsored/article/popularity-of-gaming

The Extra-Terrestrial — is so notorious that it became the symbol of the video game crash in the early 1980s.[2] Post-mortems of *E.T.*'s failure point to several causes, not the least of which its poor design. By *design*, we mean the various elements of the game experience: art, graphics, game mechanics, story, and level design. The focus of this paper is on the latter (and is defined in more detail below). In particular, we focus on *single-player* games where game designers must pay careful attention to curating the experience of their players (as opposed to multi-player games where players can interact with one another to generate experiences). The focus on single-player games is well-justified. Despite the ready connectivity of today's gaming world, single-player experiences remain one of the most popular gaming segments on consoles, PCs, and mobile devices.[3]

Single-player video games can be very long experiences, sometimes lasting dozens of hours. Accordingly, games must be broken down into "sessions" that can be consumed in one sitting. The most common form of "sessioning" in games is by the notion of levels. A *level* is a discrete unit of gameplay with a beginning, middle, and end that moves forward the game's story and often introduces new obstacles or game mechanics.[4] Level design concerns finding the right balance of game elements and sequencing them to make the level engaging and satisfying.

Some practicing game designers have proposed the use of optimization tools to assist in designing levels. Paul Tozour, an experienced game designer, wrote about the challenge in an article for Gamasutra, a leading video game design website at the time.[5] As stated in this article, a major consideration when designing a level is balancing "reward" and "difficulty" in the arrangement of game elements. To make things concrete, consider the design of a side-scrolling action game like Capcom's classic *Mega Man 2*. Each level consists of platforming sections (i.e., sections that involve skilled jumping), standard enemy encounters, and one or more "boss" (i.e., difficult) enemy encounters. Standard and boss enemy encounters test the player's strategy and reflexes while platforming sections serve as tests of dexterity and hand-eye coordination. Different types of encounters also net different rewards. Defeating standard enemies may offer much-needed boosts to health or

[2] https://www.npr.org/2017/05/31/530235165/total-failure-the-worlds-worst-video-game

[3] In a recent report, Sony revealed that single-player experiences are more popular than multi-player experiences on the Sony PlayStation platform (https://www.vice.com/en/article/5dp34k/internal-sony-docs-explain-how-activities-became-a-cornerstone-for-ps5). Large game developer EA also reports robust sales of single-player games in 2021 across all platforms (https://www.pcgamer.com/singleplayer-games-live-service/). The slate of best-selling games during 2020 on Steam (the predominant delivery platform for games on PC) features numerous single-player games (Assasin's Creed Odyssey, Uncharted 4, Horizon: Zero Dawn, Cyberpunk 2077, etc.) (https://store.steampowered.com/sale/BestOf2020). A recent market research report shows that single-player mobile games remain the most popular single segment of the video game industry (https://www.limelight.com/resources/white-paper/state-of-21online-gaming-2019).

[4] The use of the word 'level' here should not be confused with the notion of experience level or skill level of players. 'Level' here exclusively refers to a discrete "chunk" of gameplay.

[5] https://gamasutra.com/blogs/PaulTozour/20131201/206006/Decision_Modeling_and_Optimization_in_Game_Design_Part_9_Modular_Level_Design.php

ammunition, while boss fights may earn the player new weapons or unlock new areas for exploration. While *Mega Man 2* is a classic video game from the 1980s, the challenge of level design is as relevant today as it ever was. Many popular games are still designed in the mold of classics like *Mega Man 2* (including the *Lego Star Wars* series, the *New Super Mario Bros.* series, and *Minecraft Dungeons*), while other large "open-world games" offer "story mission" components with a sequential level-based structure (such as *Cyberpunk 2077* and the *Assassin's Creed* series).

This paper takes up this basic level design question, as proposed by Tozour and others, to sequence a set of given game elements (obstacles, enemy encounters, puzzles, etc.) to form an enjoyable player experience. The question of designing game elements is equally as interesting but beyond the scope of our study here. The assumption that game elements are given and then assembled into levels is consistent with game design practice. Consider, for example, the classic video game *Mario Brothers 3* by Nintendo that contains multiple "worlds" that consist of themed collections of levels with similar enemies and encounter styles. The enemies and encounter types are designed at the "world" level, whereas individual levels within the world sequence these enemies and encounter types. See Tozour (2013) for further discussion.

There are by now canonical level designs in video games. An intuitive design is one of increasing difficulty and reward as the level proceeds. As the player meets earlier tests they are more prepared to tackle later challenges. However, there is also logic for a U-shaped design where levels start difficult, become easier, then crescendo towards a difficult finish. This design was not uncommon in coin-operated video game arcades, where having a rapid succession of failed attempts could drive up revenue. Social pressure and bragging rights among arcade patrons can drive players to "overcome" the initial challenge, only to be rewarded by a section of the game that is easier to handle, leading up to a "boss" of monumental difficulty.

Another classical design for console action games (like *Megaman 2* described above) is the "mini-boss-end-boss" structure, where levels start out easy, reach a peak of tension in the middle of the level with a "mini-boss" encounter, then easing off before another crescendo to an even more difficult "end-boss" encounter. Other level designs resemble more of a workout routine: starting easy (warm-up) and ending easy (cool down) with an intermediate peak of difficulty.

Our research question is simple: under what conditions are these qualitatively different level designs optimal? The "conditions" refer to the nature of the game elements themselves, namely their rewards and difficulties. Our analysis reveals that differences in the reward-to-difficulty ratio lead to qualitatively different optimal level designs. The notion of optimality is that of maximizing the player's experienced utility accrued up to the end of the level. This objective reflects the fact that player satisfaction is experienced dynamically throughout the level and is assessed when the player decides whether or not to continue the game upon the completion of a level.

In order to answer this research question, we develop an optimization model for deciding the sequence of a given set of game elements to optimize the experienced utility of the player taking into account three psychological factors: accomplishment adaptation, stress adaptation, and memory decay. Accomplishment adaptation refers to the process by which utilities from rewards wane as players become accustomed to them.[6] Stress adaptation refers to how disutility for expending effort diminishes as players become accustomed to certain challenges. This phenomenon is well-understood by game designers. Players can adapt to difficulty quickly as they become accustomed to challenges (Kalmpourtzis 2018, Schell 2019). Memory decay refers to the psychological fact that people tend to put more emphasis on recent experiences than older experiences.[7] The limits of attention and memory capacity have been identified as a key component in understanding game design. For example, in Section 27 of Part 6 of Hiwiller (2015), an examination of how appropriate design needs to consider the limited memory capacity of players. In Chapter 4 of Hodent (2017), it describes the theory of memory loss in detail with the concept of forgetting curves.

Other authors have studied related research questions leading to optimization problems with a similar structure. The most related papers to ours are the seminal (Das Gupta et al. 2016) and related papers (including Roels (2019, 2020), Li et al. (2022)) that study the optimal design of experiential services considering both memory decay and adaptation to rewards. These papers find U-shaped and so-called IU-shaped structures for service quality against time. While our method of analysis draws much inspiration from these papers, our model and results are different. Most critically, difficulty and stress are essential characteristics of the video game experience that are not considered in these previous models. Video games are not passive and so it is not a surprise that models that assume a passive consumer (as in Das Gupta et al. (2016), Roels (2019), Li et al. (2022)) do not suffice to capture the tradeoffs that interest us. Indeed, we find many level designs (like the classical "mini-boss-end-boss" design) that are not predicted by existing models.

As for our findings, we analyze our proposed mathematical model of optimal level design to characterize when different qualitative designs are optimal. Our strongest analytical results are in the case when reward and difficulty are proportional; that is, easy game elements give small rewards while hard game elements give large rewards. This is common in the design of individual game elements, as this is consistent with the psychological theory of "flow" championed by psychologist Csikszentmihalyi (1990), whose ideas have significant influence among video game designers[8] and academic researchers of video game design (see, for instance, Cowley et al. (2008)).

[6] Studies from the psychology literature that examine and measure the adaptation process are referenced in detail in Das Gupta et al. (2016), Li et al. (2022).

[7] Studies from the psychology literature that discuss memory decay are also explored at length in Das Gupta et al. (2016), Li et al. (2022).

[8] See for instance this article on Gamasutra on the concept of flow: `https://www.gamasutra.com/view/feature/166972/cognitive_flow_the_psychology_of_.php`

In the theory of flow, the difficulty and reward for experiences should be balanced to help the participant achieve “optimal” experience called *flow*. If an activity requires little effort, an outsized reward feels hollow and unearned. Meanwhile, a task that is very difficult but reaps little reward leads to frustration. In the “sweet spot” of flow, the participant feels sensations of timelessness, happiness, and acute focus. Indeed, video games are often cited as an example of an experience highly adept at achieving “flow” in players, something of concern to parents, policy-makers, and researchers (see, for instance, the review article Kuss and Griffiths (2012)).

A practical implication of Csikszentmihalyi’s theory is that “flow” is best achieved when rewards and difficulty are proportional. It turns out that a key analytical driver of results in our model is precisely the reward-to-difficulty ratio. For example, when the proportion of reward-to-difficulty is high, the optimal design mimics that of more passive experiences like that studied in Das Gupta et al. (2016). This is intuitive since when difficulty is low, the gaming experience is not unlike a passive service experience. Classic games like *Dragon’s Lair* — which is essentially an animated movie with very simple interactive elements separating scenes — is an example of a game with a high reward-to-difficulty ratio.

Intermediate cases give rise to the possibility of optimal “mini-boss, end-boss”-like designs — what we call N-shaped designs since the difficulty, in this case, follows an ‘N’-shaped pattern (see Figure 1 for an illustration). The intuition here is that a crescendo of sustained difficulty from the beginning of the level to the end builds up too much stress in the player, which can negatively impact their remembered utility given memory decay. Instead, the design starts with a crescendo of difficulty and rewards, so that the player adjusts to difficulty slowly and diminishes the amount of disutility accrued due to stress. Once accustomed to a certain level of difficulty at the peak of the crescendo (where the “mini-boss” is encountered), the remaining pattern is similar to a pure entertainment experience with a U-shaped design. The diminuendo subsequence in the middle of the level serves to reset the reference point for rewards and helps the player relax. The final crescendo sequence helps to create a grand finale experience accentuated in the memory of the player. Our analysis shows that such designs are optimal under easy-to-accept assumptions of player behavior. Our model can also pinpoint when and how to place a peak.

We also show conditions under which an *inverted* N-shaped design is optimal. This is the case when players adapt more quickly to difficulty than to rewards. Such a setting can prevail in “serious” games designed for educational and training purposes. Here, players expect the game to be challenging and are in a mood to learn and adapt to difficulty, but unlike inverted U-shaped designs where rewards are very low, rewards for difficulty are significant enough so that, at the outset of the level, high rewards get the player “going” with positive reinforcement. We see these types of designs in educational games like the mathematics-based role-playing game *Prodigy*.

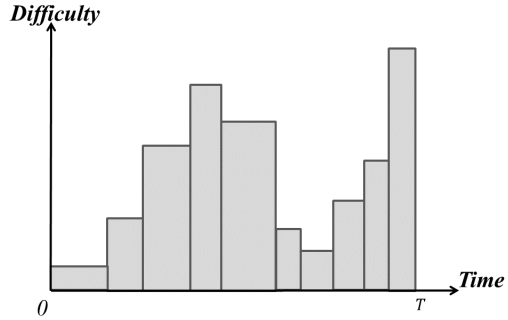


Figure 1 The N-shaped Game Design

To further our analysis of the structure of optimal level designs, we undertake a thorough numerical study in Section 5, which allowed us to explore other reward structures (other than proportional) and some additional structures of the optimal level designers. There, we show that N-shaped designs perform much better than naive strategies in many scenarios. We also find that the most difficult "boss" game elements are most commonly placed at the ends of levels, even under very general reward and difficulty inputs. We also show that the distance between the "boss" and the next hardest element (the "mini-boss") depends on the associated rewards. The outcome follows a pattern of "separated gains" and "integrated losses", as studied in Thaler (1985), Thaler and Johnson (1990).

In summary, this paper provides practical guidance to game designers on how to match level design to the relationship between reward and difficulty inherent in their game's mechanics. We make the following contributions:

- To our knowledge, this is the first paper to introduce a formal mathematical model for solving the sequencing problem inherent in video game level design.
- We provide mathematical justification for common level designs seen in practice, including N-shaped level designs, showing that they are optimal under certain conditions. Previous models for studying the design of experiential services (such as Das Gupta et al. (2016), Roels (2019), Li et al. (2022)) are unable to justify the optimality of these types of designs.
- We incorporate behavioral elements into our models that are acknowledged as being significant by game designers in a way that is mathematically elegant and tractable. This includes behavioral elements not studied in the literature on the design of experiential services.
- We show that the essence of our findings is robust to generalizations that add mathematical complexity at the cost of tractability but also capture more general game design scenarios.

It is worth noting that whereas our focus is on the design of single-player games, our analysis applies to other interactive service settings where agents must make effort in the course of receiving a service. Examples include designing trails in an outdoor adventure park, structuring the activities in a drop-in dance class, or scheduling activities at a summer camp for children. To make this concrete, consider the summer camp example. Activities of a summer camp take different amounts of effort for children to participate in and have different rewards. The overall goal is maximizing the remembered enjoyment of the campers so that they may return customers for the next summer. Designers of a plan of activities at a summer camp may use some of the insights of this study to sequence and structure these experiences in light of the accomplishment and stress processes that we identify in our study of games.

The paper is organized as follows. Section 2 summarizes related work on video games and the design of experiential services. Section 3 presents our main mathematical model of level design that is grounded in the behavioral theories of reward-seeking, difficulty aversion, and memory loss. Section 4 presents our main theoretical findings, including characterizations of when U-shaped, inverted U-shaped, N-shaped, and inverted N-shaped designs are optimal. Section 5 provides a thorough numerical exploration that provides additional insights. Section 6 concludes the main body of the paper. The e-companion has the following content. Appendix A contains all technical proofs of results in the main body. Appendix B provides tables that summarize our main analytical findings. Appendix C provides an integer programming formulation for our level design problem used in our numerical study. Appendix D gives a full specification of the parameters for an illustrative example that appears in the paper. Appendix E considers an extension to our setting where game elements can be repeated as a robustness check to our main insights. Appendix F uses real data from the game *Mario Maker 2* to illustrate how our model can be calibrated in practice. Appendix G provides a description of our model in the general setting of interactive services, using the design of a summer camp for children as an illustrative example.

2. Related Work

This paper is related to two burgeoning streams of research in operations management, information systems, and marketing. The first is on business and design questions motivated by the video game context. Many of these papers are motivated by a similar central question — how game design relates to player engagement, retention, and monetization? — but none specifically look at the question of level design. The second stream of research in on designing services and work routines that take into consideration customer and worker behavior. These papers form the main methodological inspiration for our work.

Research primarily motivated by video games is a new and rapidly developing area in business research, crossing the disciplinary boundaries of operations management, information systems, and marketing. This includes research on the design of in-game advertising (Turner et al. 2011, Guo et al. 2019b, Sheng et al. 2022), the design of virtual currency systems (Guo et al. 2019a, Meng et al. 2021), and the sale of virtual items (Huang et al. 2020, Jiao et al. 2020, Runge et al. 2021, Chen et al. 2021, Vu et al. 2020).

We mention three papers in the video game literature that are arguably the most related to the current study. Huang et al. (2020) study how the concept of player "engagement" can be used to improve the design of games (specifically in protocols for matching players), leading to increased play and improved revenues. Sheng et al. (2022) also formalize the concept of engagement in a dynamic model for determining the optimal deployment of revenue-generating in-game advertising. (The concept of engagement in video games is also studied by Huang et al. (2019).) Ascarza et al. (2020) conduct a large-scale field experiment to draw empirical connections between game difficulty and player retention. All three studies examine the connection between game design and player motivation. Ascarza et al. (2020) relate these concepts to the notion of the difficulty of a game.

In a high-level sense, our work also relates to player motivation and progression, but with a different lens. While the target practical audience of the previous papers might be those at game companies working on the business side of revenue generation, our focus here is to provide tactical insights to "frontline" design staff in charge of structuring game content. We consider the issue of engagement in the design unit of a "level" and ask what we can do to maximize the utility of the player (a proxy for engagement) given a set of more granular design elements. In this sense, we build on the findings of previous research — on the importance of engagement in games — and move towards tactical level-design questions.

Our work also builds on a growing literature concerned with behavioral aspects of offering experiential services based on the seminal work of Das Gupta et al. (2016). Connections between our work and this literature were already described at some length in the introduction, so we will not belabor the connection here. We would be remiss, however, not to mention important empirical work in the OM literature on experiential service design (notably Dixon and Verma (2013), Dixon et al. (2017), Dixon and Victorino (2019)) that has played an important complementary role to the development of optimization models like Das Gupta et al. (2016), Roels (2019), Li et al. (2022), often providing insights that inform and enrich modeling choices. These empirical studies investigate the sequence of service elements and their relationship with behaviors like surprise and anticipation using experiments. In these experiments, the elements only have a single property (service level), whereas the elements in our model have two properties: reward and difficulty. In addition, the prescriptions of these empirical studies and our model's prescriptions differ. Based on empirical findings, Dixon and his collaborators recommend U-shaped and crescendo designs and find no empirical evidence that prescribes N-shaped designs, which can be optimal in our model. This arises from the two factors in our model and shows how our results contrast with the prescriptions found in the literature based on experiments.

A related line of inquiry into the design of experiential services is research into how worker fatigue impacts the optimal design of training and work regimens. Although fatigue is a classical topic in the study of operations management, only recently has fatigue been studied from a mathematical optimization perspective. We mention two papers that share common attributes with our current study, namely in how their analysis must tackle two "factors" impacting their objective functions analogous to our accomplishment and stress processes.

The first paper is Baucells and Zhao (2019) which looks at how fatigue impacts disutility and productivity of workers in a continuous-time framework. While the two factors of disutility and productivity are similar to our accomplishment and stress processes, the analysis in Baucells and Zhao (2019) is different and leads to different conclusions. In particular, the optimization problem they study allows for a continuous choice of worker effort, while our optimization is constrained to a given set of game elements. Moreover, Baucells and Zhao (2019) find that the optimal design of effort is one of increasing or U-shaped effort profiles, while our models reveal the optimality of N-shaped difficulty designs seen in practice.

The second paper is Roels (2020), which studies how to optimally design a training regimen to optimize performance on some target date. Intense training contributes to two factors: fitness and fatigue. These factors are somewhat analogous to our accomplishment and stress processes, but with some important differences. First, there is a single driver (intensity) behind both fitness and fatigue in Roels (2020), whereas in our study accomplishment is driven by rewards, and stress is driven by difficulty. This leaves open the possibility that rewards and difficulty are not perfectly correlated. However, the more significant difference between the analysis of Roels (2020) and our study is in terms of the objective function. Roels (2020) aims to optimize fitness at a given point in time (a deadline) whereas we look at remembered utility accumulated throughout the time horizon. This is an important distinction that also informs our selection of related interactive service examples in the introduction. There, we mentioned summer camps and dance classes as settings where our model applies, but did not offer training programs and rehabilitation programs, which are a better fit with the model in Roels (2020). The reason is that the goal of summer camps and dance classes are more about creating a great experience for repeat customers (that fits our model), whereas training and rehabilitation programs have a "fitness" or "readiness" goal by the end of the program (that fits the model in Roels (2020)).

Concluding our comparison with Roels (2020), we note that our analysis has more in common with the "one-factor" model in Das Gupta et al. (2016) than the "two-factor" model of Roels (2020).

Despite the differences mentioned above, Roels (2020) does find N-shaped and, more generally, wave-like patterns of intensity that mirror some of our results, but in this different context.

It is also worth mentioning related literature in theoretical information economics that was initiated by Ely et al. (2015), with a growing literature of applications and extensions (see, for instance, Nalbantis and Pawlowski (2019), Buraimo et al. (2020), Renault et al. (2017)). The major distinction between this line of research and the work following Das Gupta et al. (2016) is that the former focus on "forward-looking" behavioral concepts like "suspense" and "surprise", while the latter tends to focus on behaviors that are backward-looking. Forward-looking concepts have the added analytical complication of tracking beliefs, something we feel complicates the tradeoffs of interest in the current study.

Finally, we note that sequencing "jobs" in an operational setting (including sequencing work on machines and assembly lines or sequencing surgeries in operating rooms) is a classical problem in the operations management literature with papers that use linear programming (LP) and integer programming (IP) techniques to solve scheduling problems that date back to the time of some of the earliest developments in LP and IP (for example, Bowman (1959), Bakshi and Arora (1969), Emmons (1969)) up to more recent developments (for example, Naderi et al. (2021), Meng et al. (2020)). This direction of research differs from ours in several ways, but the most salient aspect is the difference in objectives. The focus of the job sequencing literature is to minimize measures of run times like "tardiness" and/or fixed costs due to setups from switching between jobs. The utility of neither the worker (which is often envisioned as a machine) nor the customer are typically considered. In contrast, sequencing in level design (and service design more broadly) naturally focuses attention on maximizing customer utility.

3. Model

A game designer seeks to optimally sequence a collection of n given game elements into a level that maximizes the satisfaction of a representative player. As described in the introduction, a level is a discrete "chunk" of gameplay that a player might tackle in a single "session" of play. Game elements refer to incremental units of a video game level, including enemy encounters, puzzles, or obstacles like a set of platforms to traverse. Each game element $i \in [n] \triangleq \{1, 2, \ldots, n\}$ has an associated reward r_i, a fixed duration τ_i, and a difficulty level d_i. The reward r_i can represent in-game "loot" that the player unlocks when passing game element i, or some more psychological notion of utility experienced by the player associated with the "fun" of the element or sense of accomplishment in completing it. The duration τ_i is the expected time it takes to pass game element i. The difficulty level d_i indicates how much mental and physical energy a player exhausts to pass game element i. It is important to emphasize that the game elements and their data (rewards, difficulties, and duration) are all given.

The game designer selects a permutation π of the set $[n]$ where $\boldsymbol{\pi} = (\pi(1), \ldots, \pi(n))$ where $\pi(i)$ is the ith game element in the sequence. For example, if there are three game elements indexed by the set $\{1,2,3\}$ then the sequence $\boldsymbol{\pi} = (2,1,3)$ designs the level with game element 2 first, followed by game element 1, and finally, game element 3. We assume that the elements are indexed in increasing order of rewards; that is $r_1 \le r_2 \le \cdots \le r_{n-1} \le r_n$.

We consider a level design problem with a fixed duration $T = \sum_{i=1}^n \tau_i$. We denote by $t_{\pi(i)} = \sum_{j=1}^i \tau_{\pi(j)}$ the completion time of game element $\pi(i)$, and by $\bar{t}_{\pi(i)} = \sum_{j=i}^n \tau_{\pi(j)}$ the duration from the starting time of game element $\pi(i)$ until the end of the level. Observe that $T = \bar{t}_{\pi(i)} + t_{\pi(i)} - \tau_{\pi(i)}$. For simplicity of notation, we omit π in the subscripts and use $u_{(i)}$, $d_{(i)}$, $\tau_{(i)}$, t_i, and $\bar{t}_i$ to represent $r_{\pi(i)}$, $d_{\pi(i)}$, $\tau_{\pi(i)}$, $t_{\pi(i)}$ and $\bar{t}_{\pi(i)}$, respectively. Figure 2 provides a graphical representation of this notation.

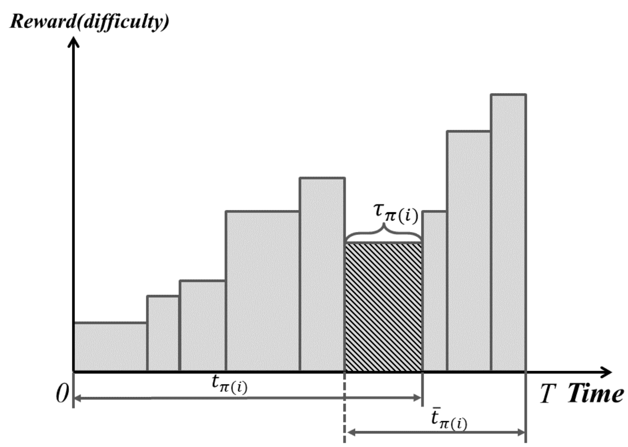


Figure 2 **Time Intervals for game element** $\pi(i)$**:** $\tau_{\pi(i)}, \bar{t}_{\pi(i)}, t_{\pi(i)}$.

3.1 Gameplay Satisfaction

We adopt a framework similar to Das Gupta et al. (2016) and Li et al. (2022) to quantify the player's retrospective perception of a level as the remembered utility accumulated from time 0 to time T. Our expression for this remembered utility draws on three psychological concepts, namely, (a) accomplishment adaptation, (b) stress adaptation, and (c) memory decay. These concepts reflect three typical behaviors in gameplay: reward-seeking, difficulty aversion, and memory loss.

The accomplishment process reflects the player's passion for seeking rewards. While people are attracted by the sensation of "winning", players also experience negative feelings during gameplay. If the game is difficult, players can come to feel anxious or frustrated, particularly when exposed to extended durations of difficulty (Chen 2007). Accordingly, we introduce a stress process that reflects the dynamics of a player's aversion to difficulty.

We introduce a memory decay process to reflect the player's memory loss due to the player's limited ability to remember what happened during the experience of a level. We must therefore examine a player's remembered utility of a level when assessing his appreciation of the design. Table 1 summarizes the relationship between the psychological process and the player behavior in the gameplay.

Table 1 Psychological Process and Player Behavior

Psychological Process	Player Behavior	Outcome
Accomplishment process	Reward seeking	Utility
Stress process	Difficulty aversion	Disutility
Memory decay process	Memory loss	Remembered utility

To formalize the accomplishment and stress processes, we follow the adaptation model of Aflaki and Popescu (2013) in a similar pattern to Das Gupta et al. (2016), where experienced utility and disutility are functions of deviations from a reference point and this reference point evolves according to a differential equation akin to Newton's law of cooling. In our model, the accomplishment process is the source of utility and the stress process is the source of disutility. Each of these processes evolves according to its own adaptive process with given parameters. These two processes are described in the next two subsections.

As the player has both positive and negative feelings from gameplay, utility from rewards and disutility from difficulty jointly affect the game experience. Following our description of the accomplishment and stress processes, we combine them to determine a (net) utility process in Section 3.1.3. At time t during the player's experience of the level, we determine an instantaneous (net) utility at time t by subtracting the instantaneous disutility from the instantaneous utility at time t. In doing so, we adopt a linear-form model similar to Roels (2020), which is based on the athletic performance model by Banister et al. (1975) with two effects, one positive (fitness) and one negative (fatigue). While performance is not the subject of our study, we believe a linear-form model is justified here because it similarly weighs two psychological impacts on utility, one positive (reward) and one negative (difficulty). This matches with game design theory, which states that the game should balance its reward scheme and the degree of challenge (Schell 2019).

For the memory decay process, we consider a memory decay process with exponential memory decay as Das Gupta et al. (2016) and Li et al. (2022). The memory decay process determines the relative weight of each game element and converts the instantaneous utility to remembered utility. Therefore, the sequence of game elements will affect the perception of the game experience.

Combining the three psychological effects, we present the framework of our study in Figure 3.

3.1.1 The Accomplishment Process. The accomplishment process reflects the psychological phenomenon of reward-seeking and adaptation to rewards. On the one hand, players prefer to

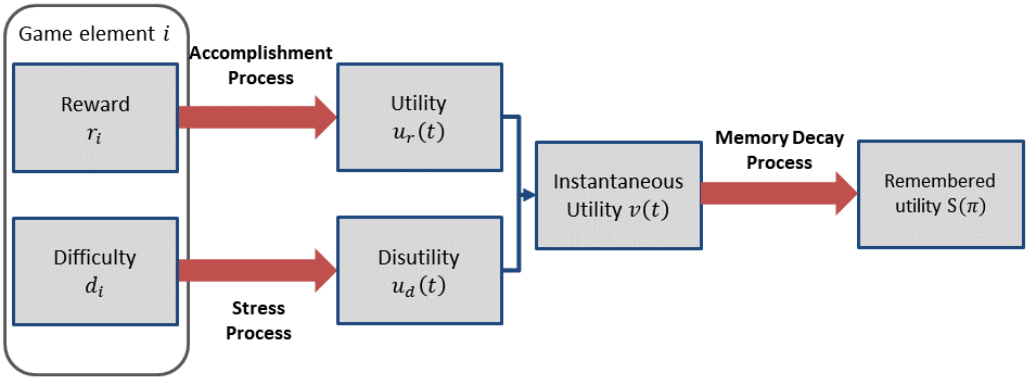


Figure 3 **The Combining Effects of Accomplishment, Stress, and Memory Decay**

receive more rewards, on the other hand, they gradually adapt to the gain and seek greater rewards (Plass et al. 2015).

To model the accomplishment process, we follow the adaptation model of Aflaki and Popescu (2013). For a given schedule $\boldsymbol{\pi}$, we denote by $f_{\boldsymbol{\pi}}(t)$ the player's reference reward at time t. For simplicity of notation, we omit π in the subscript and use $f(t)$. The instantaneous utility experienced at time $t \in [t_{i-1}, t_i]$ is a function of the difference between the current reward and the reference reward, which is given by:

$$u_r(t) = U_r\left(r_{(i)} - f(t)\right), \tag{1}$$

where $U_r(\cdot)$ is the player's utility function for rewards. We assume that utility function $U_r(\cdot)$ is linear, such that $U_r\left(r_{(i)} - f(t)\right) = u_{r,0} + a\left(r_{(i)} - f(t)\right)$, where $u_{r,0}$ is the initial utility from the experience and a is a coefficient. We can normalize $u_{r,0}$ to 0, and a to 1 without loss by simply rescaling utilities (recall that utilities are only defined up to affine scaling, see Chapter 1 of (Mas-Colell et al. 1995)).

We assume that the rate of change of the reference reward is proportional to the instantaneous utility $u_r(t)$; i.e., the rate of change of reference reward $f(t)$ at time $t \in [t_{i-1}, t_i]$ is:

$$\frac{df(t)}{dt} = \alpha u_r(t) = \alpha\left(r_{(i)} - f(t)\right),$$

where we refer to $\alpha > 0$ as the *degree of reward-seeking* of the player. Parameter α depicts the speed of adaptation to rewards. The larger is the risk-seeking degree α, the faster the reference reward accumulates. Players with very large α have insatiable appetites for rewards, even as they earn rewards they require even greater rewards to stay happy.

The reference reward at time $t \in [t_{i-1}, t_i]$ for a player with risk-seeking degree α is:

$$f(t) = r_{(i)} - \left(\left(r_{(1)} - f(0)\right) + \sum_{j=2}^{i}\left(r_{(j)} - r_{(j-1)}\right)e^{\alpha t_{j-1}}\right)e^{-\alpha t}. \tag{2}$$

With (1) and (2), the utility at $t \in [t_{i-1}, t_i]$ can be expressed as:

$$u_r(t) = \left(\left(r_{(1)} - f(0) \right) + \sum_{j=2}^{i} \left(r_{(j)} - r_{(j-1)} \right) e^{\alpha t_{j-1}} \right) e^{-\alpha t}. \tag{3}$$

3.1.2 The Stress Process. The stress process reflects the psychological phenomenon of difficulty aversion and adaptation. A game is not an unbroken sequence of rewards. Effort must be exerted in order to earn rewards and this effort is proportional to the difficulty of the game element. We assume that players adapt to difficulty analogously to how they adapt to rewards. This fits the common understanding of game design, which suggests that players learn from playing and find that challenge diminishes when faced with equally difficult game elements (Kalmpourtzis 2018, Schell 2019).

As illustrated by Figure 3, the stress process governs disutility due to effort exerted in overcoming difficulty. For a given schedule $\boldsymbol{\pi}$, we denote by $g_{\boldsymbol{\pi}}(t)$ the player's reference difficulty at time t. For simplicity of notation, we often omit π in the subscripts and use $g(t)$ instead. The disutility at time $t \in [t_{i-1}, t_i]$ is a function of the difference between the current difficulty and the reference difficulty, which is given by:

$$u_d(t) = U_d \left(d_{(i)} - g(t) \right), \tag{4}$$

where $U_d(\cdot)$ is the disutility function. As before, we assume that U_d is linear and let $U_d(d-g) = \delta(d-g)$, where δ is a given positive constant. We scale δ to 1 without loss for simplicity of the analysis. For completeness, we verify this assertion in Lemma EC.9 in Appendix A.[9]

Same as the accomplishment process, we assume the rate of change in the reference difficulty is proportional to the disutility at time t; that is, the change rate of the reference difficulty $g(t)$ at time $t \in [t_{i-1}, t_i]$ is:

$$\frac{dg(t)}{dt} = \beta u_d(t) = \beta \left(d_{(i)} - g(t) \right),$$

where $\beta > 0$ (with $\beta \neq \alpha$) is the *degree of difficulty-aversion*. Parameter β depicts the speed of adaptation to difficulty. The larger is the difficulty-aversion degree β, the faster the reference difficulty accumulates.

The reference difficulty at time $t \in [t_{i-1}, t_i]$ is:

$$g(t) = d_{(i)} - \left(\left(d_{(1)} - g(0) \right) + \sum_{j=2}^{i} \left(d_{(j)} - d_{(j-1)} \right) e^{\beta t_{j-1}} \right) e^{-\beta t}. \tag{5}$$

With (4) and (5), the disutility at $t \in [t_{i-1}, t_i]$ can be expressed as:

$$u_d(t) = \left(\left(d_{(1)} - g(0) \right) + \sum_{j=2}^{i} \left(d_{(j)} - d_{(j-1)} \right) e^{\beta t_{j-1}} \right) e^{-\beta t}. \tag{6}$$

[9] Arguing for a simple normalization of utility without loss does not suffice here because we have already executed a normalization of the utilities for rewards in the previous subsection. This is why we introduce a secondary argument for why we may assume $\delta = 1$ without loss found in Lemma EC.9.

3.1.3 The Memory Decay Process. The memory decay process reflects the psychological phenomenon of memory loss and it converts instantaneous utility into remembered utility. This works on the (net) utility derived from the instantaneous utility from rewards and disutility from difficulty. Instantaneous utility is affected by both the accomplishment and the stress processes.

As shown in Figure 3, we assume the instantaneous utility experienced at time $t \in [t_{i-1}, t_i]$ is a function of the utility from rewards and disutility from difficulty given by

$$v(t) \triangleq V(u_r(t), u_d(t)), \tag{7}$$

where $V(\cdot)$ is an aggregate utility function over utilities u_r and disutilities u_d.

We assume that the utility function $V(\cdot)$ is linear with $V(u_r(t), u_d(t)) = v_0 + \delta_r u_r(t) - \delta_d u_d(t)$, where v_0 is the initial instantaneous utility and $\delta_r, \delta_d > 0$ are given coefficients of the utility. Same as before, we normalize v_0 to 0 and scale δ_r and δ_d to 1 without loss for simplicity of the analysis. This is verified in Lemma EC.9 in Appendix A.

We assume that the player has an exponential memory decay process with rate $\gamma > 0$ and $\gamma \neq \alpha, \beta$. Then the player's cumulative remembered utility $S(\boldsymbol{\pi})$ is given by:

$$S(\boldsymbol{\pi}) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} v(t) e^{-\gamma(T-t)} dt, \tag{8}$$

where $v(t)$ is as defined in (7). Therefore, the latest game element will weigh more when players recall the game journey.

Combining (3), (6), (7), and (8) shows that the remembered utility of a level can be expressed as:

$$\begin{aligned} S(\boldsymbol{\pi}) = &\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\left(r_{(1)} - f(0)\right) e^{-\alpha t} + \sum_{j=2}^{i} \left(r_{(j)} - r_{(j-1)}\right) e^{-\alpha\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt \\ &- \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\left(d_{(1)} - g(0)\right) e^{-\beta t} + \sum_{j=2}^{i} \left(d_{(j)} - d_{(j-1)}\right) e^{-\beta\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt. \end{aligned} \tag{9}$$

We assume that the player does not have any previous gameplay experience, so that $f(0) = 0$ and $g(0) = 0$. Let $r_{[0]} = 0$ and $d_{[0]} = 0$. Continuing from (9) we have:

$$\begin{aligned} S(\boldsymbol{\pi}) = &\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\sum_{j=1}^{i} \left(r_{(j)} - r_{(j-1)}\right) e^{-\alpha\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt \\ &- \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\sum_{j=1}^{i} \left(d_{(j)} - d_{(j-1)}\right) e^{-\beta\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt, \\ = &\sum_{i=1}^{n} \left(r_{(i)} - r_{(i-1)}\right) \frac{e^{-\alpha \bar{t}_i} - e^{-\gamma \bar{t}_i}}{\gamma - \alpha} - \sum_{i=1}^{n} \left(d_{(i)} - d_{(i-1)}\right) \frac{e^{-\beta \bar{t}_i} - e^{-\gamma \bar{t}_i}}{\gamma - \beta}, \\ = &\sum_{i=1}^{n} r_{(i)} \left(\frac{e^{-\alpha \bar{t}_i} - e^{-\gamma \bar{t}_i}}{\gamma - \alpha} - \frac{e^{-\alpha \bar{t}_{i+1}} - e^{-\gamma \bar{t}_{i+1}}}{\gamma - \alpha} \right) - \sum_{i=1}^{n} d_{(i)} \left(\frac{e^{-\beta \bar{t}_i} - e^{-\gamma \bar{t}_i}}{\gamma - \beta} - \frac{e^{-\beta \bar{t}_{i+1}} - e^{-\gamma \bar{t}_{i+1}}}{\gamma - \beta} \right). \end{aligned} \tag{10}$$

3.2 The Level Design Problem

In this section, we formulate the level design problem. To simplify the expression of game satisfaction, we first introduce the function $\Phi(t|\theta,\gamma)$, where θ can be either the risk-seeking degree α or the difficulty-aversion degree β. The function $\Phi(t|\theta,\gamma)$ is given by:

$$\Phi(t|\theta,\gamma)=\frac{e^{-\theta t}-e^{-\gamma t}}{\gamma-\theta}, \tag{11}$$

where $\theta \neq \gamma$, $\theta,\gamma>0$, and $\Phi(t|\theta,\gamma)\geq 0$.

It is straightforward to see that $\Phi(t|\theta,\gamma)$ is continuous and twice differentiable in t. As shown in Lemma EC.1 in Appendix A, $\Phi(t|\theta,\gamma)$ is a concave-convex function with one inflection point and one stationary point. Let $T_0(\theta,\gamma)$ be the inflection point and $T_0'(\theta,\gamma)$ be the stationary point, whose formulation is shown in (EC.1) in Appendix A. These inflection and stationary points are important indicators of structure discussed later in Theorem 3 and Propositions EC.1 and EC.2.

By (10) and (11), we can formulate the level design problem as:

$$\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi})=\sum_{i=1}^{n} r_{(i)}\left(\Phi(\bar{t}_i|\alpha,\gamma)-\Phi(\bar{t}_{i+1}|\alpha,\gamma)\right)-\sum_{i=1}^{n} d_{(i)}\left(\Phi(\bar{t}_i|\beta,\gamma)-\Phi(\bar{t}_{i+1}|\beta,\gamma)\right). \tag{12}$$

In Appendix G we demonstrate how this model can apply to interactive service problems that go beyond the video game context.

4. Optimal Structure of Game Design

We now examine the structural properties of optimal solutions to (12). This section contains two subsections. The first considers the special case where rewards and difficulties are proportional. As mentioned in the introduction, this is consistent with the concept of "flow" and is a common design principle in video games (Chen 2007). In the second subsection, we examine the case of the more general reward and difficulty patterns.

4.1 Sequencing Game Elements with Proportional Reward

In this section, we consider the case that the reward is proportional to the difficulty of the game element, with a uniform reward ratio $k>0$, where $r_i=kd_i$.

Under proportional rewards, the player's remembered utility with proportional reward can be expressed by:

$$\begin{aligned}S(\boldsymbol{\pi})&=\sum_{i=1}^{n} kd_{(i)}\left(\Phi(\bar{t}_i|\alpha,\gamma)-\Phi(\bar{t}_{i+1}|\alpha,\gamma)\right)-\sum_{i=1}^{n} d_{(i)}\left(\Phi(\bar{t}_i|\beta,\gamma)-\Phi(\bar{t}_{i+1}|\beta,\gamma)\right),\\&=\sum_{i=1}^{n} d_{(i)}\left(\left(k\Phi(\bar{t}_i|\alpha,\gamma)-\Phi(\bar{t}_i|\beta,\gamma)\right)-\left(k\Phi(\bar{t}_{i+1}|\alpha,\gamma)-\Phi(\bar{t}_{i+1}|\beta,\gamma)\right)\right).\end{aligned}$$

To simplify the expression, we define the function

$$\Psi\left(t|\alpha,\beta,\gamma,k\right)=k\Phi\left(t|\alpha,\gamma\right)-\Phi\left(t|\beta,\gamma\right).$$

Using this notation, we rewrite the level design problem with proportional reward (LDPP) as

$$\max_{\boldsymbol{\pi}} S\left(\boldsymbol{\pi}\right)=\sum_{i=1}^{n} d_{(i)}\left(\Psi\left(\bar{t}_i|\alpha,\beta,\gamma,k\right)-\Psi\left(\bar{t}_{i+1}|\alpha,\beta,\gamma,k\right)\right). \tag{LDPP}$$

It is straightforward to see that $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is continuous and twice differentiable in t. As shown in Lemmas EC.2 and EC.3 in Appendix A, we prove that $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ can have one or two of inflection point(s) and one or two stationary point(s).

Let $T_1\left(\alpha,\beta,\gamma,k\right)$ and $T_2\left(\alpha,\beta,\gamma,k\right)$ be the inflection points when there are two points, and $T_2\left(\alpha,\beta,\gamma,k\right)$ be the unique inflection point when there is only one inflection point. For simplicity, we will drop the arguments and use T_1 and T_2 when there is no possibility for confusion. For example, in Theorems 1 and 2, we find that the player prefers either a crescendo or diminuendo subsequence within $[0,(T-T_2)^+]$, $[(T-T_2)^+,(T-T_1)^+]$, and $[(T-T_1)^+,T]$.

Our analysis suggests that the player's tastes are influenced by the joint effects of the parameters α, β, γ and k. To express the structural property of the optimal solution, we define the two thresholds

$$\underline{k}\triangleq\begin{cases}\frac{\beta+\gamma}{\alpha+\gamma}, & \text{if } \alpha>\beta,\\ \frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha<\beta \text{ and } \alpha>\gamma,\\ 0, & \text{if } \alpha<\beta \text{ and } \alpha<\gamma\end{cases} \quad\text{and}\quad \overline{k}\triangleq\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha>\beta \text{ and } \beta>\gamma,\\ +\infty, & \text{if } \alpha>\beta \text{ and } \beta<\gamma,\\ \frac{\beta+\gamma}{\alpha+\gamma}, & \text{if } \alpha<\beta.\end{cases}$$

First, we describe properties of the optimal sequence when the game's duration is sufficiently long (i.e., $T>T_2$).

Theorem 1. *When the game duration is sufficiently long (i.e., $T>T_2$), in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP, the elements' rewards (difficulties) are in the following structure.*

(i) When $k\le\underline{k}$, the optimal structure is an inverted U-shaped sequence.

(ii) When $\underline{k}<k\le\overline{k}$, there are two cases of the optimal structure.

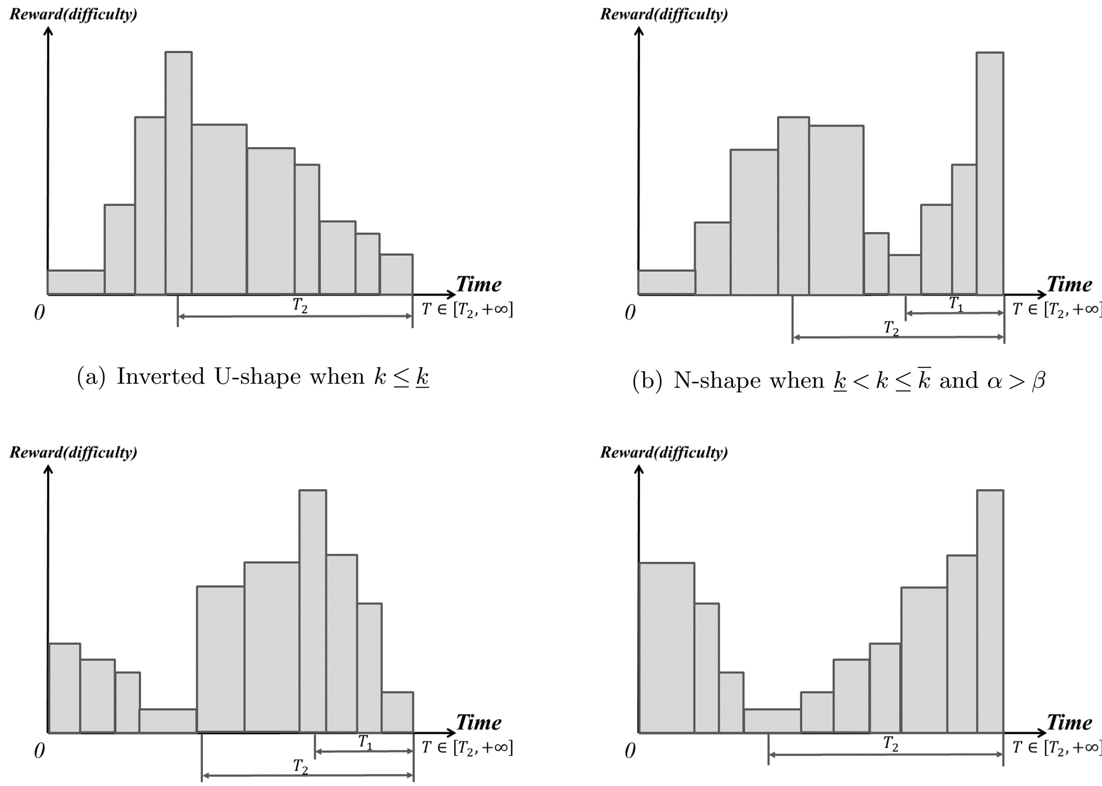
(iia) If $\alpha>\beta$, the optimal structure is a N-shaped sequence.

(iib) If $\alpha<\beta$, the optimal structure is an inverted N-shaped sequence.

(iii) When $k>\overline{k}$, the optimal structure is a U-shaped sequence.

We present the optimal structures in Figure 4 and summarize the mathematical expressions of the optimal structure in Table EC.1 in Appendix B. Theorem 1 is arguably the central result of the paper, and so we deliberate on its meaning in the next few paragraphs.

In case (i), the rewards are so low in proportion to difficulty that utilities from the stress adaptation process dominate the accomplishment process. Here, we need to think about managing the



(a) Inverted U-shape when $k \leq \underline{k}$

(b) N-shape when $\underline{k} < k \leq \overline{k}$ and $\alpha > \beta$

(c) Inverted N-shape when $\underline{k} < k \leq \overline{k}$ and $\alpha < \beta$

(d) U-shape when $k > \overline{k}$

Figure 4 An illustration of the optimal structures of the LDPP.

stress of the player, so a huge jump in difficulty will cause a lot of disutility, so we have a warm-up and cool-down to avoid jumps. In other words, when the reward ratio is low (i.e., $k < \underline{k}$), the problem will become a workout design problem, whose optimal structure is an inverted U-shaped sequence regardless of the values of α and β. This is easy to understand. When you do a workout, the player gets tired very easily. We see this game design in genres that require intensive body movement like the arcade classics *Dance Dance Revolution* or *Whack-A-Mole*.

Conversely, when the reward ratio is high (i.e., $k \geq \overline{k}$), the problem essentially becomes the design of an entertainment or service experience as studied in Das Gupta et al. (2016). Accordingly, the optimal structure follows the U-shaped pattern identified in Das Gupta et al. (2016). This optimal structure is well-suited to games in which the plot is the most important issue (e.g., interactive fiction like the classic arcade game *Dragon's Lair*).

These two extreme cases are well-covered by previous literature, while the intermediate case (ii) yields fresh insights. In case (ii), rewards and difficulties are roughly even in weight (with k between $\underline{k}$ and $\bar{k}$) and so the degrees of reward-seeking α and difficulty aversion β start to play a pivotal role. Because this "second-order affect" has bite, we no longer see the "extreme" cases of U-shaped and inverted U-shape. (We will see the even more extreme designs of pure crescendo and diminuendo arise in short duration levels in Theorem 2.) Indeed, Case (iia) yields an N-shaped design that starts with a preliminary crescendo of difficulty followed by a U-shaped finish. Case (iib) has an inverted N-shaped design that starts with a diminuendo of difficulty followed by an inverted U-shaped finish. In both of these cases, we see a more even distribution of hard and easy elements, common to many popular video games. Let's examine these two subcases in turn.

Case (iia) is distinguished by game designs with similar rewards and difficulties, but where the degree of reward-seeking outstrips the degree of difficulty aversion; that is, $\alpha > \beta$. It is our contention that $\alpha > \beta$ is a common case for players who play games largely as a form of entertainment. The player adjusts quickly to rewards and so demands increasing rewards to maintain a given level of utility. On the other hand, the players adapt slowly to difficulty and so suffer a lot of disutility if there is a sudden spike in challenge.

This is reflected in the optimal N-shaped design. The design starts with a crescendo of difficulty and rewards so that the player adjusts to difficulty slowly and diminishes the amount of disutility accrued. Once accustomed to a certain level of difficulty at the peak of the crescendo, the remaining pattern is similar to a pure entertainment experience with a U-shaped design. The diminuendo subsequence in the middle of the level serves to reset the reference point for rewards and helps the player relax. The final crescendo sequence helps create a grand ending experience accentuated in the memory of the player who otherwise adapts quickly to rewards.

By contrast, Case (iib) is distinguished by game designs with similar rewards and difficulties, but now where the degree of difficulty aversion outstrips the degree of reward-seeking; that is, $\alpha < \beta$. We believe this scenario is common in "serious" games which are played not purely for entertainment, but for educational, training, and adherence purposes (see, for example, Plass et al. (2015) and Kalmpourtzis (2018) for discussions of study games, Sardi et al. (2017) for medical programs, and Seaborn and Fels (2015) for workplace incentive programs). The online game *Prodigy* is designed for school-age children to learn mathematics in a role-playing game (RPG) style environment. In a game like *Prodigy*, players are in a learning mode (no one mistakes Prodigy for a pure entertainment game) so they can adjust quickly to difficulty, whereas they are pleasantly surprised to be getting rewards while learning math and so adjust slowly in their expectations of rewards.

The inverted N-shaped design is intuitive under these conditions. The initial diminuendo subsequence at the beginning provides the player with a spike in initial rewards, which translates into a spike of utility because adaptation to rewards is slow. On the other hand, an initial spike in difficulty that slowly diminishes is expected in an educational game whose goal is to teach a difficult topic like mathematics. Players quickly adjust to these expectations as they figure out the types of questions or problems they are being presented with. As the player moves to the later part of the level, the inverted U-shaped subsequence is reminiscent of Case (i). Players have experienced enough rewards to undertake an ascending peak of rewards and difficulties, followed by a cool down. The decrescendo at the end takes advantage of a steady decline in disutility as the game elements become easier.

It is important to appreciate the differences between Case (iia) and Case (iib). In Case (iia), ending the level with a U-shaped subsequence will create high utilities, but we need a crescendo subsequence in the beginning to let the player adapt to difficulty first. In Case (iib), difficulty plays a more important role. This time, ending the level with an inverted U-shaped sequence will create high utilities, but we need a diminuendo subsequence in the beginning to give the player an initial sense of accomplishment at the outset. This design takes advantage of their fresh mind at the outset to get some of the difficult tasks under their belt, then reset their nerves for a final inverted U-shaped push.

From both the theoretical results and the practical use, we can tell that the game designer will have to understand the difficulty of the game to make a better design. If he is designing a low-difficulty game, then he can create an experience similar to pure entertainment. If he is designing a high-difficulty game, then he can forge an experience like a workout. When the designer is designing a game with medium difficulty, then the distribution of easy and hard elements should be more balanced and follow the characteristics of the players. For games with a greater emphasis on entertainment, an N-shaped design with a mini-boss-end-boss structure is optimal. For games designed to educate or train, an inverted N-shaped design should be considered.

We complete the analysis initiated in Theorem 1 by investigating the case of short levels (i.e., when $T < T_2$).

THEOREM 2. *When the game duration is short (i.e., $T < T_2$), the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP exhibits the following structure:*

(i) When $k \leq \overline{k}$, the optimal structure degenerates to a diminuendo sequence if $T < T_2$.

(ii) When $\underline{k} < k \leq \overline{k}$, there are two cases of the optimal structure.

- *(iia) When $\alpha > \beta$, the optimal structure degenerates to a U-shaped sequence if $T_1 < T < T_2$, and a crescendo sequence if $T < T_1$.*
- *(iib) When $\beta > \alpha$, the optimal structure degenerates to an inverted U-shaped sequence if $T_1 < T < T_2$, and a diminuendo sequence if $T < T_1$.*

(iii) When $k > \underline{k}$, the optimal structure degenerates to a crescendo sequence if $T < T_2$.

The above proposition suggests that game duration is another key issue. If the game is designed with a compact duration, then you can only form part of the optimal sequence. This echoes the findings in Das Gupta et al. (2016) and Li et al. (2022) that the optimal structure may degenerate when the duration is not long enough. Crescendo and diminuendo designs are also common in games. Mobile games in the "endless runner" genre (like the popular *Jetpack Joyride*) start easy and quickly build towards greater and greater difficulty, reflecting a crescendo design. By contrast, many of the original arcade games, like *Donkey Kong*, start punishingly difficult. This reflects the different types of players that the games were designed to attract. In the arcades of the late 1970s and early 1980s, video gaming had a public and competitive feel (captured, for example, in the 2007 documentary *King of Kong: A Fistful of Quarters*). Games that presented a stern challenge were favored by players as a way to "rank" the gaming abilities of those in the arcades.

Table 2 summarizes the results in Theorems 1 and 2 on the optimal structure of levels based on our model.

Table 2 Optimal Structures of the LDPP

k	Reward Ratio	α,β	T	Duration	Optimal Structure
$k \le \underline{k}$	Low	$\alpha,\beta > 0$	$T > T_2$	Long	Inverted U-shape
			$0 < T \le T_2$	Short	Diminuendo
$\underline{k} < k \le \overline{k}$	Medium	$0 < \beta < \alpha$	$T > T_2$	Long	N-shape
			$T_1 < T \le T_2$	Medium	U-shape
			$0 < T \le T_1$	Short	Crescendo
		$0 < \alpha < \beta$	$T > T_2$	Long	Inverted N-shape
			$T_1 < T \le T_2$	Medium	Inverted U-shape
			$0 < T \le T_1$	Short	Diminuendo
$k > \overline{k}$	High	$\alpha,\beta > 0$	$T > T_2$	Long	U-shape
			$0 < T \le T_2$	Short	Crescendo

We can see that the value-to-reward ratio k, parameters α, β, and game duration T can jointly affect the optimal structure. When $T < T_2$, the optimal structure starts to degenerate. N-shaped and inverted N-shaped sequences can be optimal only when the reward ratio is in the medium level $\underline{k} < k \le \overline{k}$.

Finally, we consider the special case where reward equals difficulty (i.e., $k = 1$). In this case, we can interpret that the player accumulates a sense of accomplishment purely by the challenge of the game elements. The following corollary gives a very compact breakdown of how all six possible game designs (crescendo, diminuendo, inverted U-shape, U-shape, N-shape, inverted N-shape) are possible as the remaining parameters (besides k) change.

Corollary 1. *When the reward equals the difficulty of each game element, in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP, the elements' rewards (difficulties) are in the following structure.*

(i) When $\alpha > \beta$, the optimal structure is an N-shaped sequence if $T > T_2$, a U-shaped sequence if $T_1 < T < T_2$, and a crescendo sequence if $T < T_1$.

(ii) When $\beta > \alpha$, the optimal structure is an inverted N-shaped sequence if $T > T_2$, an inverted U-shaped sequence if $T_1 < T < T_2$, and a diminuendo sequence if $T < T_1$.

One takeaway here is how pivotal a role is played by the parameters α and β. As we discussed earlier, one can associate $\alpha > \beta$ with audiences that are looking for more entertainment experiences, while $\alpha < \beta$ is associated with more learning or training experiences. Corollary 1 highlights that these two orientations support fundamentally different optimal level designs. This is a nontrivial design insight for developers of educational games who might otherwise benchmark their level design against entertainment-focused games.

4.2 Game Design with General-reward Scheme

In this section, we consider the general case where rewards and difficulties are no longer proportional. In this case, we refer to (12) as the level design problem with general reward (LDPG). When there is no proportional relationship, we were, for the most part, only able to analyze (12) numerically as an integer optimization problem. See Appendix C for a description of the IP formulation that we worked with and see Section 5 for our numerical findings.

One special case we were able to analyze was the case where all elements share a common reward (alternatively, a common difficulty). When the elements share a common difficulty $d_i = d$ for all $i \in [n]$, the level design problem with general reward and fixed difficulty (LDPGFD) can be expressed by:

$$\begin{aligned}\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} r_{(i)} \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - \sum_{i=1}^{n} d\left(\Phi\left(\bar{t}_i|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\right), \\ &= \sum_{i=1}^{n} r_{(i)} \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - d\Phi\left(T|\beta,\gamma\right). \end{aligned} \tag{13}$$

When the elements share a fixed reward $r_i = r$ for all $i \in [n]$ the level design problem with general reward and fixed reward (LDPGFR) can be expressed by:

$$\begin{aligned}\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} r \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - \sum_{i=1}^{n} d_{(i)}\left(\Phi\left(\bar{t}_i|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\right), \\ &= r\Phi\left(T|\alpha,\gamma\right) + \sum_{i=1}^{n} d_{(i)} \left(\Phi\left(\bar{t}_{i+1}|\beta,\gamma\right) - \Phi\left(\bar{t}_i|\beta,\gamma\right)\right). \end{aligned} \tag{14}$$

In these two settings, we were able to show the following structural result. We should remark that this result can be shown via similar arguments to that found in Das Gupta et al. (2016), since once rewards or difficulties are fixed, the model effectively becomes a "single factor" model like that studied in Das Gupta et al. (2016). In order to be self-contained, we include a detailed proof in the appendix, but are careful to point out the similarities between our argument and those found in Das Gupta et al. (2016).

THEOREM 3. *Recall that $T_0(\alpha,\gamma)$ and $T_0(\beta,\gamma)$ are the unique inflection point of function $\Phi(\alpha,\gamma)$ and $\Phi(\beta,\gamma)$ respectively.*

(i) When the elements share a fixed difficulty in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGFD, the rewards are in a U-shaped sequence if $T > T_0(\alpha,\gamma)$, and a crescendo sequence if $T < T_0(\alpha,\gamma)$.

(ii) When the elements share a fixed reward in the optimal schedule $\boldsymbol{\pi}^*$ *of the LDPGFR, the difficulties are in an inverted U-shaped sequence if* $T > T_0(\beta,\gamma)$*, and a diminuendo sequence if* $T < T_0(\beta,\gamma)$*.*

While we cannot prove the properties of optimal structure for the general level design problem (12), we found some interesting results that are commonly observed in video game level designs seen in practice. We mention here one numerical instance with an optimal sequence, as illustrated in Figure 5. The parameters defining the instance can be found in Table EC.3 in Appendix D.

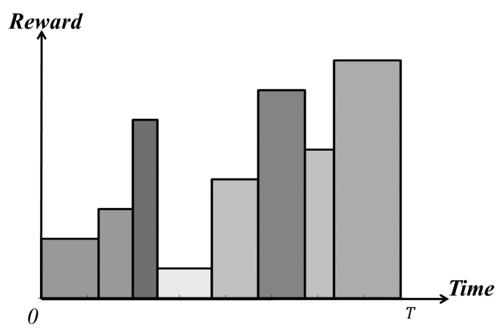


Figure 5 **An illustration of the optimal sequence of the LDPG. The heights of the shaded bars mark the rewards of the game elements and the gray-scale shadings represent the difficulty of the game elements.**

In Figure 5, the heights of the shaded bars mark the rewards of the game elements and the gray-scale shadings represent the difficulty of the game elements. The height of the bars indicates the size of the reward, while darker shades correspond to more difficulty.

In this example, the optimal schedule exhibits a "wave-like" structure. Difficulty increases for a certain period (e.g., the first three elements in Figure 5), then the game turns easy in a short time, followed by another crescendo subsequence of difficulty (e.g., the middle three elements in Figure 5). To match the changes in difficulty, the reward sequence also follows a wave-like structure. This design pattern matches recommendations by game designers like Hiwiller (2015) and Hodent (2017), and service designers like Lawrence (2014), which indicate that a structure with multiple peaks and drops is preferred by the players and customers.

We may develop an intuition for how the wave structure arises via the discussion that follows Theorem 1. Peaks in difficulty are followed by a "cooling" period to slow the stress process. Rewards also work in patterns of crescendos and diminuendos so that players do not become "numb" to high rewards by adjusting their expectations. Players look for a challenging and rewarding experience, making intermittent crescendos of difficulty and reward attractive, but an ever-increasing crescendo makes players increasingly stressed at the same time of becoming inured to the rewards. An example of a popular game with this "wave-like" pattern of difficulty and rewards is the *Plants vs Zombies* series of mobile games. In the games in this series, the player makes defenses using plants to ward off waves of attacking zombies. Zombies come in waves of varying difficulties.

Table 3 summarizes the optimal structures and the conditions, and we summarize the mathematical expressions of the optimal structure in Table EC.2 in Appendix B.

Table 3 Optimal Structures of the LDPG

Problem	Situation	T	Duration	Optimal Structure
LDPGFD	Fixed difficulty	$T > T_0$	Long	U-shape
		$0 < T \leq T_0$	Short	Crescendo
LDPGFR	Fixed reward	$T > T_0$	Long	Inverted U-shape
		$0 < T \leq T_0$	Short	Diminuendo
LDPG	General	$T > 0$	Any duration	Wave-like

As a final note, we have included the description of a related model in an appendix of the paper (see Appendix E). One of the distinguishing features of games is that elements are virtual, meaning that they can be reproduced costlessly multiple times within a level. This is in contrast with service design problems, like those studied in Das Gupta et al. (2016), where repeating a service element may be costly or not possible.

Here the decision space is extended to allow the level designer to choose the number of each game element to deploy (within a given time limit) as well as how to sequence these elements. This is related to, but different than, the challenge of choosing the duration of elements studied in Das Gupta et al. (2016). In Appendix E, we study the optimal structure of the final sequence of game elements (allowing for repeats) in both the proportional reward and general reward settings. Our results in this setting are consistent with the findings in the base model we study in this section. Accordingly, these results should be viewed as a robustness check for our main conclusions.

5. Numerical Study

We use numerical approaches to gain further insight into the structure of optimal level designs. We began this analysis with an illustrative result in Figure 5 in the previous section, but take a more systematic approach. We are interested in questions of the prevalence of the different optimal level designs (U-shaped, N-shaped, crescendo, etc.) across many instantiations. In the first two subsections below, we look at structured reward and difficulty data (either proportional or other structured protocols). These results show that N-shaped optimal designs are not uncommon (Tables 4 and 7) and are often fairly well-approximated by U-shaped designs (Table 5) in remembered utility. Diminuendo, and the inverted U- and N-shaped designs are much less common in our experiments. Optimal U-shaped designs are the most common.

In a third subsection, we examine optimal level structure when rewards and difficulties of elements are unstructured. As might be predictable, the optimal design is wave-like with a high probability, mimicking what we see in Figure 5. N-shaped designs are also more prevalent than

U-shaped designs. In this section, we also investigate the question of where the boss (i.e., the most difficult element) is typically positioned and the average distance between "peaks" of waves.

In our experiments, we consider level design problems with eight elements. We generate 150 instances of rewards, difficulties, durations, and parameters α, β, and γ. For each instance in Section 5.1, we randomly generate difficulties $\boldsymbol{d}$ and reward ratios k from independent uniform distribution Uniform$(0, 10)$, and we set the reward vector as $\boldsymbol{r} = k\boldsymbol{d}$. For each instance in Section 5.2, we use the protocol-based reward as presented in Table 6 that look at structured, but non-proportional reward structures and generate random data appropriately. For the instances in Section 5.3, the rewards, difficulties, and durations are randomly generated from independent uniform distribution Uniform$(0, 10)$. As Das Gupta et al. (2016), parameters α, β, and γ are drawn from independent Gamma distribution Gamma(k_G, θ_G), with shape k_G and scale θ_G. This leads to unstructured rewards and difficulties. We conduct the numerical study under parameters $k_G \in \{1, 2, 3\}$ and $\theta_G \in \{0.125, 0.25, 0.375\}$. For all experiments, we compute the optimal sequence by solving the IP in Appendix C.

5.1 Sequencing under Proportional Reward Scheme

In this section, we analyze the optimal structure under a proportional reward scheme. We start by analyzing the distribution of optimal structures. We record the percentage of different optimal structures across 150 instances and present the results in Table 4.

Table 4 Percentage of the Optimal Structures under the Proportional Reward Scheme (%)

(k_G, θ_G)	Crescendo	Diminuendo	Inverted U shape	U shape	Inverted N shape	N shape	N shape in theory
(1,0.125)	22.67	4.67	10.00	58.67	0.67	3.33	44.00
(1,0.25)	6.67	0.67	12.00	61.33	2.67	16.67	44.67
(1,0.375)	6.67	0	12.00	64.00	4.67	12.67	34.67
(2,0.125)	2.67	0.67	10.00	68.67	2.67	15.33	38.67
(2,0.25)	0	0	19.33	56.67	1.33	22.67	41.33
(2,0.375)	0.67	0	18.67	64.67	2.67	13.33	34.67
(3,0.125)	0	0	8.67	66.67	6.67	18.00	40.67
(3,0.25)	0	0	18.00	60.00	2.67	19.33	36.00
(3,0.375)	0	0	22.67	62.00	2.00	13.33	34.67

We can see from the table, as k_G increases, less crescendo and diminuendo sequences are optimal. An explanation is that there are fewer degenerate cases as k_G grows. Crescendo and diminuendo sequences are degenerate cases, as we summarized in Table 2. The percentage of inverted and U-shaped optimal sequences is stable under different situations. There are more N-shaped optimal sequences when θ_G is of medium value, and there are only a few instances where the inverted N-shaped sequence is optimal. The last column of the table, labeled "N-shape in theory" records the percentage of instances where the parameters are such that an N-shape design is optimal if the level was sufficiently long (that is, $T > T_2$). Around 40% of instances were found to be in this category. However, since we conducted our numerical study with only eight elements, this may not be long enough to show the whole N-shaped structure in some of these instances. In these cases, the N-shaped sequence degenerates into a crescendo or U-shaped sequence, even if the parameters allow for an N-shaped structure.

Furthermore, we study the differences in remembered utility among different sequences and compare them when N-shaped sequence is optimal. Let $\boldsymbol{\pi}_i$ stands for the schedule in structure i, where $i \in \{1, \ldots, 6\}$ stands for the crescendo, diminuendo, inverted U shape, U shape, inverted N shape, and N shape respectively. Let the sequence in structure $i \in \{1, \ldots, 6\}$ that has the largest remembered utility be $\boldsymbol{\pi}_i^*$, and the optimal sequence be $\boldsymbol{\pi}^*$. Then, the optimal gap of the best solution in structure i is given by $p_i = \frac{|S(\boldsymbol{\pi}^*) - S(\boldsymbol{\pi}_i^*)|}{S(\boldsymbol{\pi}^*)} \cdot 100\%$.

Next, we report on the optimality gap of all structures for the instances where an N-shaped sequence is optimal in Table 5. We can see from the table that the optimal gap is stable with different (k_G, θ_G) for many structures. U-shaped sequence has the least gap to the N-shaped sequence, and the gap increases as k_G grows. The optimal U-shaped sequence thus acts as a reasonable heuristic, but the average gap is still considerable, often larger than 5%.

Table 5 **Average Gap of Sequences in Different Structures When N-shaped Sequence is Optimal (%)**

(k_G, θ_G)	Crescendo	Diminuendo	Inverted U shape	U shape	Inverted N shape	N shape
(1,0.125)	112.39	122.24	108.58	5.54	89.93	0
(1,0.25)	170.17	104.91	89.95	7.33	80.23	0
(2,0.125)	108.90	99.76	84.55	8.01	72.60	0
(2,0.25)	97.55	107.68	89.82	8.94	87.08	0

5.2 Sequencing under Protocol-Based Reward Scheme

Of course, we would like to go beyond proportional rewards. Instead of going immediately to completely general reward structures (which we take up in the next subsection), here we restrict to nonproportional, but structured protocols for rewards and difficulties. Our main question is whether N-shaped designs remain a prevalent design in these more general scenarios. These protocols can be thought of as broad strategies to enhance the engagement of players at the game element design stage. Table 6 illustrates an example of the protocols we consider.

Table 6 **Reward Protocols**

No.	Reward	Difficulty
1	$\boldsymbol{r} = (1/4, 1, 9/4, 4, 25/4, 9, 49/4, 16)^{\mathrm{T}}$	$\boldsymbol{d} = (1, 2, 3, 4, 5, 6, 7, 8)^{\mathrm{T}}$
2	$\boldsymbol{r} = (16, 49/4, 9, 25/4, 4, 9/4, 1, 1/4)^{\mathrm{T}}$	
3	$\boldsymbol{r} = (16, 9, 4, 1, 1/4, 9/4, 25/4, 49/4)^{\mathrm{T}}$	
4	$\boldsymbol{r} = (1/4, 9/4, 25/4, 49/4, 16, 9, 4, 1)^{\mathrm{T}}$	

In protocol 1, the reward increases as the difficulty grows, but it follows a nonlinear relationship $r_i = d_i^2/4$, rather than a linear relationship with the difficulty as in the proportional case. In protocol

2, the reward decreases as the difficulty grows. The rewards are in a U-shaped sequence in protocol 3, and an inverted U-shaped sequence in protocol 4.

We investigate the structure of the optimal sequence under the same environment as Section 5.1. However, because we consider a non-proportional reward scheme, we cannot identify the structure of the sequence based on either the reward or the difficulty. Instead, we analyze the reward ratios of a sequence of the elements. Let the reward ratio of element i be q_i given by $q_i = r_i/d_i$.

We now present the percentage of different reward ratio sequences in Table 7. Protocol 1 is the only protocol that supports inverted U-shaped sequences. Protocol 1 is closest to the proportional reward scheme, and this is the reason why it supports most of the optimal structures proposed in Section 4.1. Protocol 2 only supports the crescendo, U-shaped, and N-shaped sequence, but not the wave-like sequence. One explanation is that the highest-difficulty elements have the least rewards, and this reduces the number of wave-like sequences because reward and difficulty are negatively correlated. Protocol 3 supports more U-shaped sequences and less N-shaped sequences than protocol 4. The relative proportion of the optimal sequence is stable as θ_G changes, but there are less crescendo and diminuendo sequences when k_G is large. Similar to Section 5.1, there may be fewer degenerate instances, and hence there are fewer crescendo and diminuendo sequences. All of these demonstrate how patterns in the rewards and difficulties leads to related structures in the optimal level design.

Table 7 Optimal Reward Ratio Sequences under Different Protocols (%)

(k_G, θ_G)	Protocol	Crescendo	Diminuendo	Inverted U shape	U shape	Inverted N shape	N shape	Wave like
(1,0.125)	1	27.33	0.67	3.33	61.33	0	6.00	1.33
	2	32.00	0	0	68.00	0	0	0
	3	3.33	0	0	30.00	0	15.33	51.33
	4	1.33	0	0	11.33	0	31.33	56.00
(1,0.25)	1	10.00	2.00	14.00	57.33	0	14.00	2.67
	2	12.67	0	0	87.33	0	0	0
	3	0	0	0	26.00	0	10.00	64.00
	4	0	0	0	9.33	0	26.67	64.00
(2,0.125)	1	4.67	0	6.00	66.00	0	21.33	2.00
	2	2.67	0	0	97.33	0	0	0
	3	0	0	0	20.67	0	1.33	78.00
	4	0	0	0	13.33	0	27.33	59.33
(2,0.25)	1	1.33	0	31.33	45.33	0	13.33	8.67
	2	0	0	0	100.00	0	0	0
	3	0	0	0	18.67	0	0	81.33
	4	0	0	0	4.00	0	37.33	58.67

5.3 Sequencing under General Reward Scheme

In this section, we analyze the optimal sequence under a general reward scheme. Following the definition of reward ratio $q_i = r_i/d_i$, we start by presenting the distribution of optimal reward ratio structures in Table 8.

Table 8 Percentage of the Optimal Reward Rate Structures under the General Reward Scheme (%)

(k_G, θ_G)	Crescendo	Diminuendo	Inverted U shape	U shape	Inverted N shape	N shape	Wave like
(1,0.125)	0	0	0	0.67	3.33	5.33	90.67
(1,0.25)	0	0	0.67	0	2.0	3.33	94.00
(2,0.125)	0	0	0	0	3.33	1.33	95.33
(2,0.25)	0	0	0.67	0.67	0.67	2.67	95.33

From the table, we see that most of the instances have wave-like structure in the reward ratios. There are more inverted N-shaped and N-shaped sequences than inverted U-shaped and U-shaped sequences. No crescendo or diminuendo sequences are found. There are more wave-like sequences when k_G is moderate while θ_G does not appear to have any systematic impact.

Because of the prevalence of wave-like results in the simulations, the rest of our numerical investigations explore some of the salient features in the "waves" that we see. One obvious feature is the location of the largest "peak". This corresponds to asking about the location of the "boss" of the level. The most common design in practice is to find the "boss" at the end of the level, and so we explore how prevalent this is in our optimal level designs. Another natural question is when "peaks" (bosses and mini-bosses) are batched close together in the "waves" or spaced farther apart. This tells us something about the tempo of difficulty in the optimal level design.

To investigate this, we simplify things by isolating attention to one boss and one mini-boss in our simulations. Based on this, we consider four configurations. Let element $n-1$ be the mini-boss, and element n be the boss. The setup of the rewards and difficulties of the bosses are presented in Table 9, where the parameters x and y in the reward columns vary in $\{1,\ldots,10\}$. We randomly generate 150 instances of the difficulty and reward of the rest nonboss elements with Uniform$(0,10)$, and we consider an identical duration with $\tau = 5$ for all the elements, which facilitates our analysis of the optimal positions of the bosses.

Table 9 Configurations of the Numerical Study

Configuration	Boss		Mini-boss	
	Reward	Difficulty	Reward	Difficulty
1	$r_n = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$	$d_n = 15$	$r_{n-1} = 13$	$d_{n-1} = 13$
2a	$r_n = 15$	$d_n = 15$	$r_{n-1} = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_{n-1}$	$d_{n-1} = 13$
2b	$r_n = 60$	$d_n = 15$	$r_{n-1} = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_{n-1}$	$d_{n-1} = 13$
3	$r_n = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$	$d_n = 15$	$r_{n-1} = (0.8 + 0.2 \cdot e^{y-5}) \cdot d_{n-1}$	$d_{n-1} = 13$

In the first study, we investigate the optimal position of the boss (i.e., element n). To better present the trend of the change in the optimal position of the boss, we follow configuration 1 in Table 9. The distribution of the optimal positions of the boss is presented in Figure 6. From the figure, we see that the distribution of the optimal boss position changes as x increases. The optimal positions are evenly distributed when x is small. As x increases, there are more instances where the boss is placed in the final slot. Because the reward increases as x grows, the result suggests that the boss with higher reward is optimal to be scheduled at the end of the game. The benefit

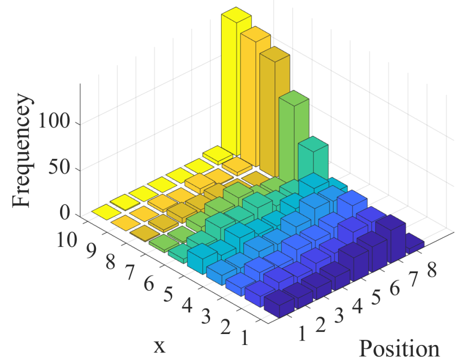
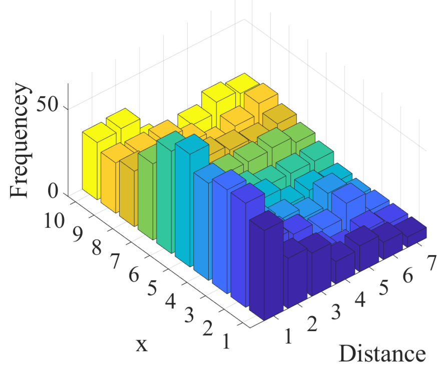


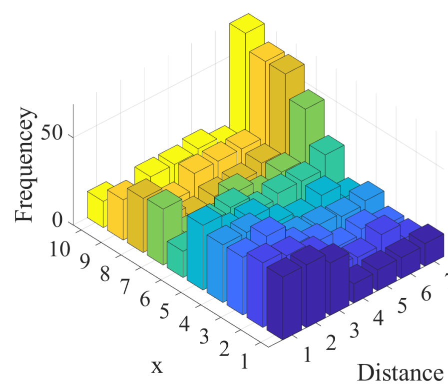
Figure 6 Optimal position of the boss as r_n changes, where $r_n = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$.

is that a boss with higher reward scheduled at the end can provide higher remembered utility for the players. Placing the most influencing element at the end is a common phenomenon studied in the literature (e.g., Kahneman et al. 1993, Das Gupta et al. 2016).

We now study the distance between the boss and mini-boss. We consider two cases listed as configuration 2a and 2b in Table 9: (i) when $r_n = 15$ and (ii) when $r_n = 60$. We record the distance between the bosses under these two cases and present them in Figure 7.



(a) When $r_n = 15$



(b) When $r_n = 60$

Figure 7 Distance between the bosses as r_{n-1} changes, where $r_{n-1} = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$

We can see from the figure that the distribution of the distance is different when $r_n = 15$ and when $r_n = 60$. When $r_n = 15$ and x is small, there are more instances with small distances. When $r_n = 60$ and x is large, there are more instances with larger distances. This result echoes the study of Thaler (1985) and Thaler and Johnson (1990), which reveal that people prefer separate gains and integrated losses. The intuition behind the result is that it is better to have wonderful moments separated to enjoy all of them, and to integrate unhappy moments to minimize pain.

Based on the previous results, we extend the investigation to the average distance between the bosses. We follow configuration 3 in Table 9 and conduct the numerical study with different values of r_{n-1} and r_n. The distribution of average distances is shown in Figure 8.

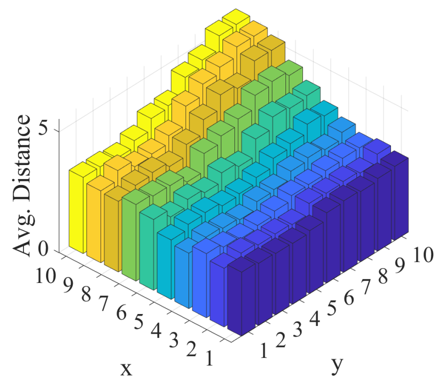


Figure 8 **Average distance between the bosses as r_n and r_{n-1} changes, where $r_n = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$ and $r_{n-1} = (0.8 + 0.2 \cdot e^{y-5}) \cdot d_{n-1}$.**

From the figure, we can see that the average distance increases as x (r_n) and y (r_{n-1}) increase. The case with $x, y = 10$ has largest average distance. The result suggests that it is optimal to separate the bosses when the reward rates of the bosses are high. The separate-boss solution provides more remembered utility because it takes advantage of the accomplishment, stress, and memory decay processes. Same as before, the results are consistent with the study on the spread effect (e.g., Loewenstein and Prelec 1993, Dixon and Verma 2013).

In this section, we have performed numerical simulations to get further insight into optimal level design structure. There is also a natural question about how a game designer can calibrate the parameters of our model (α, β, and γ) based on available game data. We describe how to perform this calibration using real data from the game *Mario Maker 2* in Appendix F.

6. Conclusion

In this paper, we presented a mathematical model to analyze the problem of designing video game levels for players who are reward-seeking, difficulty averse, and suffer from memory decay. Our analysis shows that the relative strengths of these factors, and the properties of the game elements used to sequence a level, give rise to a variety of different level designs. Appendix B summarizes these findings in two convenient tables.

We believe that future research into level design can further explore some of the complexities that we see in practice but are beyond the scope of the current model. First, in this model, we have assumed that players assess utility in a backwards-looking manner at the end of the level. This is consistent with the experiential services literature initiated by Das Gupta et al. (2016), but alternative "forward-looking" models (like those found in Ely et al. (2015)) offer other modeling opportunities to examine the optimal structure of video game levels. It would be interesting to see if these alternative theoretical foundations could provide additional insight into why certain level designs are prevalent in practice.

Our model assumes that players "stick around" until the end of the level before deciding whether or not to continue playing the game. We did this for tractability purposes, because otherwise we would need to track some "forwarding looking" information about what the player thinks will happen later when deciding if to quit a level mid-stream. We believe an extension that incorporates quitting behavior would be a major contribution, since retention of players is a core concern of game design, particularly in free-to-play games.

Some of the results we have may have promise for understanding the design of games in the "endless runner" genre, typified by the high revenue-generating *Jetpack Joyride* on mobile platforms. In endless runners, levels are procedurally generated (meaning generated randomly as they are encountered) and, in principle, have no end (hence the adjective "endless"). An infinite horizon dynamic model would be needed to study this problem, but we believe many of the insights we have developed here would be applicable in this setting, particularly the notion of how "peaks" and "valleys" of difficulty manage reward-seeking and difficulty-aversion behaviors of players.

Finally, there are applications of game design that extend beyond the classical entertainment setting studied here. The concept of gamification — using games to help people learn or comply with medical regimes, for example — is a growing area of application (see, for example, (Plass et al. 2015, Kalmpourtzis 2018, Sardi et al. 2017, Seaborn and Fels 2015)). We expect this trend to continue as games become more widely accepted as a form of meaningful interaction in society.

References

Aflaki S, Popescu I (2013) Managing retention in service relationships. *Management Science* 60(2):415–433.

Ascarza E, Netzer O, Runge J (2020) The twofold effect of customer retention in freemium settings. *Harvard Business School Working Paper 21-062* .

Bakshi MS, Arora SR (1969) The sequencing problem. *Management Science* 16(4):B–247.

Banister EW, Calvert TW, Savage MV, Bach T (1975) A systems model of training for athletic performance. *Aust J Sports Med* 7(3):57–61.

Baucells M, Zhao L (2019) It is time to get some rest. *Management Science* 65(4):1717–1734.

Bowman EH (1959) The schedule-sequencing problem. *Operations Research* 7(5):621–624.

Buraimo B, Forrest D, McHale IG, Tena J (2020) Unscripted drama: soccer audience response to suspense, surprise, and shock. *Economic Inquiry* 58(2):881–896.

Chen J (2007) Flow in games (and everything else). *Communications of the ACM* 50(4):31–34.

Chen N, Elmachtoub AN, Hamilton ML, Lei X (2021) Loot box pricing and design. *Management Science* 67(8):4809–4825.

Cowley B, Charles D, Black M, Hickey R (2008) Toward an understanding of flow in video games. *Computers in Entertainment (CIE)* 6(2):1–27.

Csikszentmihalyi M (1990) *Flow: The Psychology of Optimal Experience* (Harper & Row, New York).

Das Gupta A, Karmarkar US, Roels G (2016) The design of experiential services with acclimation and memory decay: Optimal sequence and duration. *Management Science* 62(5):1278–1296.

Dixon M, Verma R (2013) Sequence effects in service bundles: Implications for service design and scheduling. *Journal of Operations Management* 31(3):138–152.

Dixon MJ, Victorino L (2019) The sequence of service: An affect perspective to service scheduling. *Handbook of Service Science, Volume II* 49–76.

Dixon MJ, Victorino L, Kwortnik RJ, Verma R (2017) Surprise, anticipation, and sequence effects in the design of experiential services. *Production and Operations Management* 26(5):945–960.

Ely J, Frankel A, Kamenica E (2015) Suspense and surprise. *Journal of Political Economy* 123(1):215–260.

Emmons H (1969) One-machine sequencing to minimize certain functions of job tardiness. *Operations Research* 17(4):701–715.

Guo H, Hao L, Mukhopadhyay T, Sun D (2019a) Selling virtual currency in digital games: Implications for gameplay and social welfare. *Information Systems Research* 30(2):430–446.

Guo H, Zhao X, Hao L, Liu D (2019b) Economic analysis of reward advertising. *Production and Operations Management* 28(10):2413–2430.

Hiwiller Z (2015) *Players Making Decisions: Game Design Essentials and the Art of Understanding Your Players* (New Riders).

Hodent C (2017) *The Gamer's Brain: How Neuroscience and UX Can Impact Video Game Design* (CRC Press).

Huang Y, Jasin S, Manchanda P (2019) "Level Up": Leveraging skill and engagement to maximize player game-play in online video games. *Information Systems Research* 30(3):927–947.

Huang Y, Lim KH, Lin Z (2020) Leveraging the numerosity effect to influence perceived expensiveness of virtual items. *Information Systems Research* 32(1):93–114.

Jiao Y, Tang CS, Wang J (2020) Opaque selling in player-vs-player games. *Available at SSRN 3558774* .

Kahneman D, Fredrickson BL, Schreiber CA, Redelmeier DA (1993) When more pain is preferred to less: Adding a better end. *Psychological Science* 4(6):401–405.

Kalmpourtzis G (2018) *Educational Game Design Fundamentals: A Journey to Creating Intrinsically Motivating Learning Experiences* (CRC Press).

Kuss DJ, Griffiths MD (2012) Online gaming addiction in children and adolescents: A review of empirical research. *Journal of Behavioral Addictions* 1(1):3–22.

Lawrence A (2014) Boom! wow, wow, wow, boom. *Work · Play · Experience* http://workplayexperience.blogspot.com/2006/12/boom-wow-wow-wow-boom.html. Last accessed: April 9, 2018.

Li Y, Dai T, Qi X (2022) A theory of interior peaks: Activity sequencing and selection for service design. *Manufacturing & Service Operations Management* 24(2):993–1001.

Loewenstein GF, Prelec D (1993) Preferences for sequences of outcomes. *Psychological Review* 100(1):91.

Mas-Colell A, Whinston MD, Green JR (1995) *Microeconomic Theory* (Oxford University Press, New York).

Mauro L (2019) A temporal dataset from Super Mario Maker. https://www.kaggle.com/leomauro/smmnet Retrieved June 29, 2020.

Meng L, Zhang C, Shao X, Zhang B, Ren Y, Lin W (2020) More milp models for hybrid flow shop scheduling problem and its extended problems. *International Journal of Production Research* 58(13):3905–3930.

Meng Z, Hao L, Tan Y (2021) Freemium pricing in digital games with virtual currency. *Information Systems Research* 32(2):481–496.

Naderi B, Roshanaei V, Begen MA, Aleman DM, Urbach DR (2021) Increased surgical capacity without additional resources: Generalized operating room planning and scheduling. *Production and Operations Management* .

Nalbantis G, Pawlowski T (2019) US demand for European soccer telecasts: A between-country test of the uncertainty of outcome hypothesis. *Journal of Sports Economics* 20(6):797–818.

Plass JL, Homer BD, Kinzer CK (2015) Foundations of game-based learning. *Educational Psychologist* 50(4):258–283.

Renault J, Solan E, Vieille N (2017) Optimal dynamic information provision. *Games and Economic Behavior* 104:329–349.

Roels G (2019) Optimal structure of experiential services: Review and extensions. *Handbook of Service Science, Volume II*, 105–146 (Springer).

Roels G (2020) High-performance practice processes. *Management Science* 66(4):1509–1526.

Runge J, Nair H, Levav J (2021) Price promotions for "freemium" app monetization. *Available at SSRN 3357275* .

Sardi L, Idri A, Fernández-Alemán JL (2017) A systematic review of gamification in e-health. *Journal of Biomedical Informatics* 71:31–48.

Schell J (2019) *The Art of Game Design: A Book of Lenses* (CRC press).

Seaborn K, Fels DI (2015) Gamification in theory and action: A survey. *International Journal of Human-Computer Studies* 74:14–31.

Sheng L, Ryan CT, Nagarajan M, Cheng Y, Tong C (2022) Incentivized actions in freemium games. *Manufacturing & Service Operations Management* 24(1):275–284.

Thaler R (1985) Mental accounting and consumer choice. *Marketing Science* 4(3):199–214.

Thaler RH, Johnson EJ (1990) Gambling with the house money and trying to break even: The effects of prior outcomes on risky choice. *Management Science* 36(6):643–660.

Tozour P (2013) Decision modeling and optimization in game design, part 9: Modular level design. https://www.gamasutra.com/blogs/PaulTozour/20131201/206006/Decision_Modeling_and_Optimization_in_Game_Design_Part_9_Modular_Level_Design.php.

Turner J, Scheller-Wolf A, Tayur S (2011) Scheduling of dynamic in-game advertising. *Operations Research* 59(1):1–16.

Vu D, Zhao X, Stecke K (2020) Pay-to-win in video games: Microtransactions and fairness concerns. *Available at SSRN 3658537* .

E-companion for "Optimal Level Design in Video Games"

Here is a summary of the contents of this e-companion. Appendix A contains all technical proofs of results in the main body. Appendix B provides tables that summarize our main analytical findings. Appendix C provides an integer programming formulation for our level design problem used in our numerical study. Appendix D gives a full specification of the parameters for an illustrative example that appears in the paper. Appendix E considers an extension to our setting where game elements can be repeated as a robustness check to our main insights. Appendix F uses real data from the game *Mario Maker 2* to illustrate how our model can be calibrated in practice. Appendix G provides a description of our model in the general setting of interactive services, using the design of a summer camp for children as an illustrative example.

Appendix A: Technical Proofs

To analyze the properties of the optimal structure, we first define the formulation of $T_0'(\theta,\gamma)$ and $T_0(\theta,\gamma)$:

$$
\begin{aligned}
T_0'(\theta,\gamma) &= \frac{\ln\gamma-\ln\theta}{\gamma-\theta}, \\
T_0(\theta,\gamma) &= 2\frac{\ln\gamma-\ln\theta}{\gamma-\theta}.
\end{aligned}
\tag{EC.1}
$$

We prove the property of $\Phi(t|\theta,\gamma)$ in the following lemma.

Lemma EC.1. *(i) $\Phi(t|\theta,\gamma)$ is an increasing function in t for $t\in[0,T_0'(\theta,\gamma)]$; $\Phi(t|\theta,\gamma)$ is a decreasing function otherwise.*

(ii) $\Phi(t|\theta,\gamma)$ is a concave-convex function with inflection point $T_0(\theta,\gamma)$.

Proof of Lemma EC.1. By Lemma A3 in Das Gupta et al. (2016), we have $\frac{\partial\Phi(t|\theta,\gamma)}{\partial t}\geq 0$ when $t\in[0,T_0'(\theta,\gamma)]$, and $\frac{\partial\Phi(t|\theta,\gamma)}{\partial t}\leq 0$ when $t\in[T_0'(\theta,\gamma),T]$; we have $\frac{\partial^2\Phi(t|\theta,\gamma)}{\partial t^2}\leq 0$ when $t\in[0,T_0(\theta,\gamma)]$, and $\frac{\partial^2\Phi(t|\theta,\gamma)}{\partial t^2}\geq 0$ when $t\in[T_0(\theta,\gamma),T]$. The function is a concave-convex function with inflection point $T_0(\theta,\gamma)$ and stationary point $T_0'(\theta,\gamma)$. See Figure EC.1 for an illustration. *Q.E.D.*

We then prove the property of function $\Psi(t|\alpha,\beta,\gamma,k)$ in the following two lemmas.

Lemma EC.2. *Consider α, β, γ be three (different) positive parameters, $t\geq 0$, and k be arbitrary real numbers.*

(1) Suppose $\alpha>\beta$. Define $\underline{k}=\frac{\beta+\gamma}{\alpha+\gamma}$ and $\overline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \beta>\gamma,\\ +\infty, & \beta<\gamma.\end{cases}$

(1.1) When $k\leq\underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave function.

(1.2) When $\underline{k}<k<\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex-concave function.

(1.3) When $k\geq\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex function.

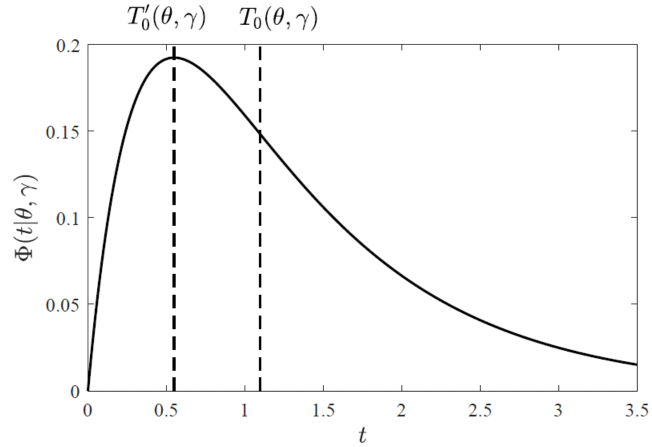


Figure EC.1 Demonstration of the function $\Phi(t|\theta,\gamma)$ when $\theta=3$ and $\gamma=1$.

(2) Suppose $\alpha<\beta$. Define $\underline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha>\gamma\\ 0, & \text{if } \alpha<\gamma\end{cases}$ *and* $\overline{k}=\frac{\beta+\gamma}{\alpha+\gamma}$.

(2.1) When $k\le\underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave function.

(2.2) When $\underline{k}<k<\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave-convex function.

(2.3) When $k\ge\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex function.

Proof. By definition, $\Psi(t|\alpha,\beta,\gamma,k)=k\Phi(t|\alpha,\gamma)-\Phi(t|\beta,\gamma)$ where $\Phi(t|\theta,\gamma)=\frac{e^{-\theta t}-e^{-\gamma t}}{\gamma-\theta}$. Then

$$\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}-\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}=k\frac{\alpha^2e^{-\alpha t}-\gamma^2e^{-\gamma t}}{\gamma-\alpha}-\frac{\beta^2e^{-\beta t}-\gamma^2e^{-\gamma t}}{\gamma-\beta}\ . \tag{EC.2}$$

Following Lemma EC.1, we know that $\Phi(t|\theta,\gamma)$ is a concave-convex function with inflection point $T_0(\theta,\gamma)=\frac{2(\ln s-\ln\gamma)}{s-\gamma}$. Specifically, $\frac{\partial^2\Phi(T_0(\theta,\gamma)|\theta,\gamma)}{\partial t^2}=0$; $\frac{\partial^2\Phi(t|\theta,\gamma)}{\partial t^2}<0$ if $t<T_0(\theta,\gamma)$; and $\frac{\partial^2\Phi(t|\theta,\gamma)}{\partial t^2}>0$ if $t>T_0(\theta,\gamma)$. We further have $\frac{\partial T_0(\theta,\gamma)}{\partial\theta}=\frac{2(1-\frac{\gamma}{s}+\ln\frac{\gamma}{s})}{(s-\gamma)^2}<0$, given that $1+\ln\xi<\xi$ for $0<\xi<1$ and $\xi>1$. Thus, $T_0(\theta,\gamma)$ decreases in θ.

We start with two special cases. First, suppose $k=0$. Then, $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=-\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}$. By Lemma EC.1, we know that $\Psi(t|\alpha,\beta,\gamma,0)$ is a convex-concave function. Second, suppose $t=T_0(\beta,\gamma)$, resulting in $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}=0$. Then, $\frac{\partial^2\Psi(T_0(\beta,\gamma)|\alpha,\beta,\gamma,k)}{\partial t^2}=k\frac{\partial^2\Phi(T_0(\beta,\gamma)|\alpha,\gamma)}{\partial t^2}$, whose sign is determined by k and $\frac{\partial^2\Phi(T_0(\beta,\gamma)|\alpha,\gamma)}{\partial t^2}$. In particular, if $\alpha>\beta$, we have $T_0(\alpha,\gamma)<T_0(\beta,\gamma)$. Following Lemma EC.1, we conclude that $\frac{\partial^2\Phi(T_0(\beta,\gamma)|\alpha,\gamma)}{\partial t^2}>0$. If $\alpha<\beta$, we have $T_0(\alpha,\gamma)>T_0(\beta,\gamma)$ and we conclude that $\frac{\partial^2\Phi(T_0(\beta,\gamma)|\alpha,\gamma)}{\partial t^2}<0$.

In the following, we focus on the case when $k\neq0$ and $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\neq0$. We rewrite (EC.2) to be

$$\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{\frac{\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}}{\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}}-\frac{1}{k}\right\}. \tag{EC.3}$$

Our goal is to determine the sign of $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}$ for any given t. We have already shown that the component $\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)$ is first negative and then positive. Below, we investigate the ratio $\frac{\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}}{\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}}$. For demonstration purposes, we denote

$$r_1(t,\alpha,\beta|\gamma)=\frac{\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}}{\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}}=\frac{\left(\frac{\alpha^2e^{-\alpha t}-\gamma^2e^{-\gamma t}}{\gamma-\alpha}\right)}{\left(\frac{\beta^2e^{-\beta t}-\gamma^2e^{-\gamma t}}{\gamma-\beta}\right)}=\left(\frac{\gamma-\beta}{\gamma-\alpha}\right)\left(\frac{\alpha^2e^{-\alpha t}-\gamma^2e^{-\gamma t}}{\beta^2e^{-\beta t}-\gamma^2e^{-\gamma t}}\right).$$

First of all, we examine its monotonicity. The first-order derivative is given by

$$\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t}=-\frac{(\beta-\gamma)e^{-(\alpha-\beta)t}\left\{\alpha^2\beta^2(\alpha-\beta)+\gamma^2[\beta^2(\beta-\gamma)e^{(\alpha-\gamma)t}-\alpha^2(\alpha-\gamma)e^{(\beta-\gamma)t}]\right\}}{(\alpha-\gamma)(\beta^2-\gamma^2e^{(\beta-\gamma)t})^2}.$$

Denote $r_2(t,\alpha,\beta|\gamma)=\alpha^2\beta^2(\alpha-\beta)+\gamma^2[\beta^2(\beta-\gamma)e^{(\alpha-\gamma)t}-\alpha^2(\alpha-\gamma)e^{(\beta-\gamma)t}]$. Clearly, $r_2(t,\alpha,\beta|\gamma)$ plays an important role in determining the sign of the derivative $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t}$. Moreover, we have

$$\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}=e^{-\gamma t}(\beta^2e^{\alpha t}-\alpha^2e^{\beta t})\gamma^2(\gamma-\alpha)(\gamma-\beta)=e^{-\gamma t}e^{\beta t}\beta^2(e^{(\alpha-\beta)t}-\frac{\alpha^2}{\beta^2})\gamma^2(\gamma-\alpha)(\gamma-\beta). \tag{EC.4}$$

It is straightforward to see from (EC.4) that there exist 6 possible scenarios as below:

(a) If $\alpha>\beta>\gamma$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first negative and then positive.
(b) If $\alpha>\gamma>\beta$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first positive and then negative.
(c) If $\gamma>\alpha>\beta$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first negative and then positive.
(d) If $\beta>\alpha>\gamma$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first positive and then negative.
(e) If $\beta>\gamma>\alpha$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first negative and then positive.
(f) If $\gamma>\beta>\alpha$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first positive and then negative.

By definition, $T_0(\alpha,\beta)$ is such that $\beta^2e^{\alpha T_0(\alpha,\beta)}-\alpha^2e^{\beta T_0(\alpha,\beta)}=0$. From (EC.4), we observe that $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}=0$ at $t=T_0(\alpha,\beta)$. Given its monotonicity, we conclude that when $t=T_0(\alpha,\beta)$, $r_2(t,\alpha,\beta|\gamma)$ reaches the lowest point in cases (a), (c) and (e); while $r_2(t,\alpha,\beta|\gamma)$ reaches the highest point in cases (b), (d), and (f). In addition, the sign of $r_2(T_0(\alpha,\beta),\alpha,\beta|\gamma)$ follows from Lemma EC.1 and the fact that $T_0(\alpha,\beta)<T_0(\beta,\gamma)$ if $\alpha>\gamma$ or $T_0(\alpha,\beta)>T_0(\beta,\gamma)$ if $\alpha<\gamma$. Specifically,

$$\begin{aligned}
r_2(T_0(\alpha,\beta),\alpha,\beta|\gamma)&=\alpha^2\beta^2(\alpha-\beta)+\gamma^2[\beta^2(\beta-\gamma)e^{(\alpha-\gamma)T_0(\alpha,\beta)}-\alpha^2(\alpha-\gamma)e^{(\beta-\gamma)T_0(\alpha,\beta)}]\\
&=\alpha^2\beta^2(\alpha-\beta)+\gamma^2[\alpha^2(\beta-\gamma)e^{(\beta-\gamma)T_0(\alpha,\beta)}-\alpha^2(\alpha-\gamma)e^{(\beta-\gamma)T_0(\alpha,\beta)}]\\
&=\alpha^2\beta^2(\alpha-\beta)+\gamma^2\alpha^2(\beta-\alpha)e^{(\beta-\gamma)T_0(\alpha,\beta)}\\
&=\alpha^2(\alpha-\beta)[\frac{\beta^2}{\gamma^2}-e^{(\beta-\gamma)T_0(\alpha,\beta)}]\\
&=\begin{cases}>0, & \text{in case (a)}\\ <0, & \text{in case (b)}\\ >0, & \text{in case (c)}\\ <0, & \text{in case (d)}\\ >0, & \text{in case (e)}\\ <0. & \text{in case (f)}\end{cases}
\end{aligned}$$

In summary, in cases (a), (c), and (e), we have shown that $r_2(t,\alpha,\beta|\gamma)$ first decreases in $t < T_0(\alpha,\beta)$ and then increases in $t \geq T_0(\alpha,\beta)$. In addition, the minimum value $r_2(T_0(\alpha,\beta),\alpha,\beta|\gamma)$ is positive. Therefore, we conclude that $r_2(t,\alpha,\beta|\gamma) > 0$ for $t \geq 0$ in cases (a), (c) and (e). On the other hand, in cases (b), (d), and (f), we have shown that $r_2(t,\alpha,\beta|\gamma)$ first increases in $t < T_0(\alpha,\beta)$ and then decreases in $t \geq T_0(\alpha,\beta)$. In addition, the maximum value $r_2(T_0(\alpha,\beta),\alpha,\beta|\gamma)$ is negative. Therefore, we conclude that $r_2(t,\alpha,\beta|\gamma) < 0$ for $t \geq 0$ in cases (b), (d), and (f).

Recall that

$$\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} = -\frac{(\beta-\gamma)e^{-(\alpha-\beta)t}}{(\alpha-\gamma)(\beta^2-\gamma^2 e^{(\beta-\gamma)t})^2} r_2(t,\alpha,\beta|\gamma).$$

We are able to make the following conclusion.

(a) If $\alpha > \beta > \gamma$, $r_2(t,\alpha,\beta|\gamma) > 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} < 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ decreases in t;

(b) If $\alpha > \gamma > \beta$, $r_2(t,\alpha,\beta|\gamma) < 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} < 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ decreases in t;

(c) If $\gamma > \alpha > \beta$, $r_2(t,\alpha,\beta|\gamma) > 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} < 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ decreases in t;

(d) If $\beta > \alpha > \gamma$, $r_2(t,\alpha,\beta|\gamma) < 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} > 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ increases in t;

(e) If $\beta > \gamma > \alpha$, $r_2(t,\alpha,\beta|\gamma) > 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} > 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ increases in t;

(f) If $\gamma > \beta > \alpha$, $r_2(t,\alpha,\beta|\gamma) < 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} > 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ increases in t;

In short, we have proven that $r_1(t,\alpha,\beta|\gamma)$ decreases in t if $\alpha > \beta$ or increases in t if $\alpha < \beta$.

Next, we explore the sign of $r_1(t,\alpha,\beta|\gamma)$. Without loss of generality, we focus on the case that $\alpha > \beta$. It is straightforward to see that $r_1(0,\alpha,\beta|\gamma) = \frac{\alpha+\gamma}{\beta+\gamma} > 0$. Second, we have

$$r_1(+\infty,\alpha,\beta|\gamma) = \left(\frac{\beta-\gamma}{\alpha-\gamma}\right) \lim_{t\to+\infty} \frac{\alpha^2 e^{(\gamma-\alpha)t} - \gamma^2}{\beta^2 e^{(\gamma-\beta)t} - \gamma^2} = \begin{cases} \frac{\beta-\gamma}{\alpha-\gamma}, & \text{in case (a): } \alpha > \beta > \gamma \\ 0, & \text{in case (b): } \alpha > \gamma > \beta \\ 0, & \text{in case (c): } \gamma > \alpha > \beta \end{cases}$$

Furthermore, when $\alpha > \beta$, we have $T_0(\alpha,\gamma) < T_0(\beta,\gamma)$. Since $r_1(t,\alpha,\beta|\gamma) = \frac{\frac{\partial^2 \Phi(t|\alpha,\gamma)}{\partial t^2}}{\frac{\partial^2 \Phi(t|\beta,\gamma)}{\partial t^2}}$, by the definitions of $T_0(\alpha,\gamma)$ and $T_0(\beta,\gamma)$ as well as Lemma EC.1, we obtain that $r_1(t,\alpha,\beta|\gamma) > 0$ when $t < T_0(\alpha,\gamma)$; $r_1(t,\alpha,\beta|\gamma) = 0$ at $t = T_0(\alpha,\gamma)$; $r_1(t,\alpha,\beta|\gamma) < 0$ when $T_0(\alpha,\gamma) < t < T_0(\beta,\gamma)$; and $r_1(t,\alpha,\beta|\gamma) > 0$ when $t > T_0(\beta,\gamma)$. Lastly, as t approaches to $T_0(\beta,\gamma)$ from the left, we have $\lim_{t\uparrow T_0(\beta,\gamma)} r_1(t,\alpha,\beta|\gamma) = -\infty$. As t approaches to $T_0(\beta,\gamma)$ from the right, we have $\lim_{t\downarrow T_0(\beta,\gamma)} r_1(t,\alpha,\beta|\gamma) = +\infty$.

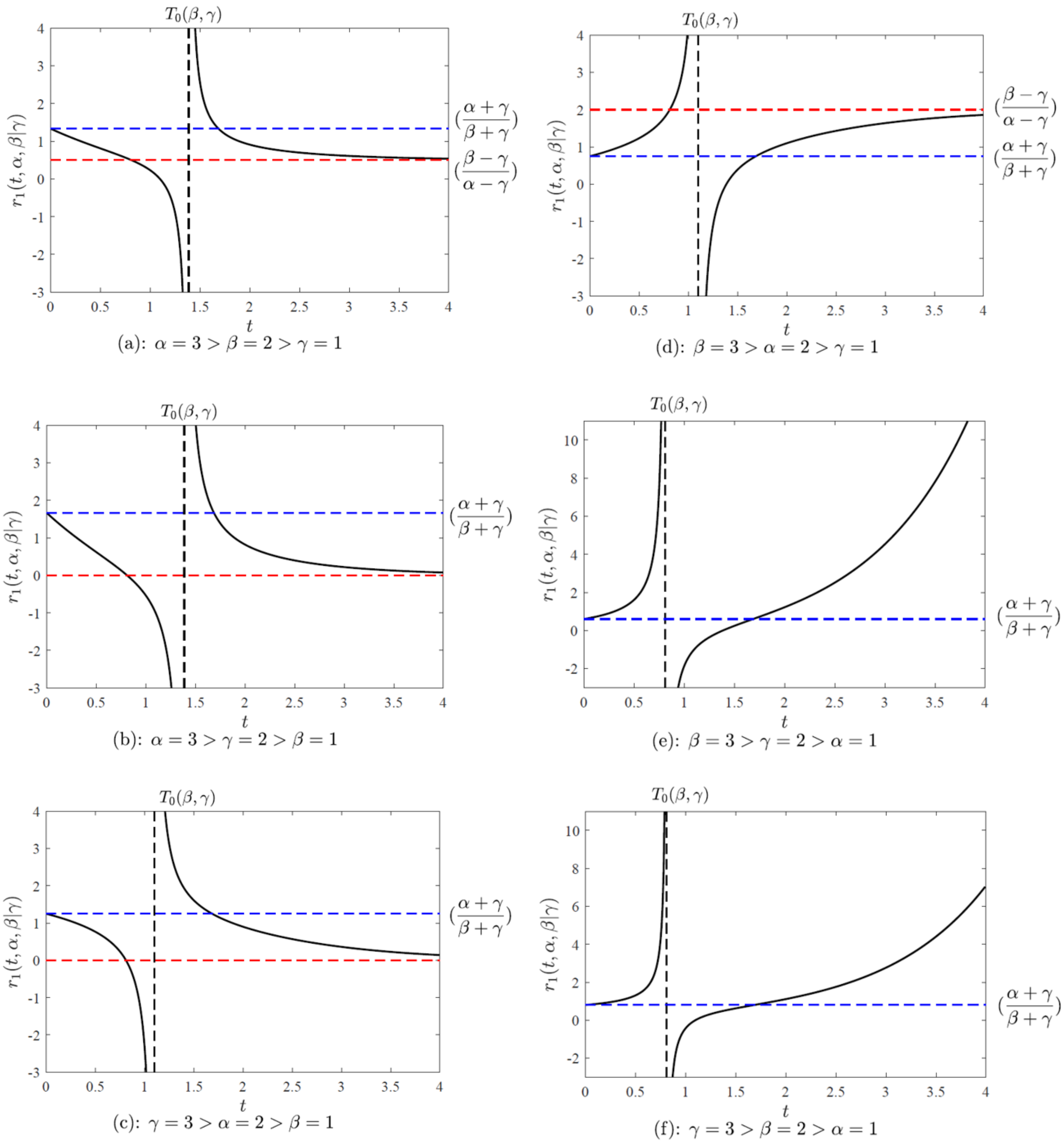


Figure EC.2 **Demonstration of the function $r_1(t,\alpha,\beta|\gamma)$. The subfigures on the left assume $\alpha>\beta$ and the subfigures on the right assume $\alpha<\beta$.**

The left column of Figure EC.2 illustrates the function $r_1(t,\alpha,\beta|\gamma)$ when $\alpha>\beta$. Following Equation (EC.3), in order to determine the sign of $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}$, we need to compare $r_1(t,\alpha,\beta|\gamma)$ with any given k. For ease of understanding, we denote $w=\frac{1}{k}$ and compare $r_1(t,\alpha,\beta|\gamma)$ with w. As suggested from Figure EC.2, we identify two thresholds for w: upper threshold $\bar{w}$ and lower threshold $\underline{w}$ where

$$\bar{w}=\frac{\alpha+\gamma}{\beta+\gamma}$$
$$\underline{w}=\begin{cases}\frac{\beta-\gamma}{\alpha-\gamma}, & \text{in case (a): } \alpha>\beta>\gamma\\ 0, & \text{in case (b) and (c): } \alpha>\gamma>\beta \text{ or } \gamma>\alpha>\beta\end{cases}$$

- If $w>\bar{w}$, the equation $r_1(t,\alpha,\beta|\gamma)=w$ has only one solution at $t>T_0(\beta,\gamma)$. Let the solution be T_1. Then $r_1(t,\alpha,\beta|\gamma)<w$ when $t<T_0(\beta,\gamma)$; $r_1(t,\alpha,\beta|\gamma)>w$ when $T_0(\beta,\gamma)<t<T_1$; and $r_1(t,\alpha,\beta|\gamma)<w$ when $t>T_1$.
- If $w\le\underline{w}$, the equation $r_1(t,\alpha,\beta|\gamma)=w$ has only one solution at $t<T_0(\beta,\gamma)$. Let the solution be T_2. Then $r_1(t,\alpha,\beta|\gamma)>w$ when $t<T_2$; $r_1(t,\alpha,\beta|\gamma)<w$ when $T_2<t<T_0(\beta,\gamma)$; and $r_1(t,\alpha,\beta|\gamma)>w$ when $t>T_0(\beta,\gamma)$.
- If $\underline{w}<w\le\bar{w}$, the equation $r_1(t,\alpha,\beta|\gamma)=w$ has two solutions. One is at $t<T_0(\beta,\gamma)$ and is denoted as T_3. The other is at $t<T_0(\beta,\gamma)$ and is denoted as T_4. Then $r_1(t,\alpha,\beta|\gamma)>w$ when $t<T_3$; $r_1(t,\alpha,\beta|\gamma)<w$ when $T_3<t<T_0(\beta,\gamma)$; $r_1(t,\alpha,\beta|\gamma)>w$ when $T_0(\beta,\gamma)<t<T_4$; and $r_1(t,\alpha,\beta|\gamma)<w$ when $t>T_4$.

Finally, recall that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ and $w=\frac{1}{k}$ by definition. In addition, $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}<0$ when $t<T_0(\beta,\gamma)$; $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}=0$ when $t=T_0(\beta,\gamma)$; and $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}>0$ when $t>T_0(\beta,\gamma)$. From the above analysis, we make the following conclusion:

- If $k<0$, which implies that $w<\underline{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is positive when $t<T_2$; it is zero when $t=T_2$; and it is negative when $t>T_2$. Thus, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave.
- If $k=0$, as mentioned at the beginning of the proof, $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}$ is convex-concave.
- If $0<k<\frac{\beta+\gamma}{\alpha+\gamma}$, which implies that $w\ge\bar{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is positive when $t<T_1$; it is zero when $t=T_1$; and it is negative when $t>T_1$. Thus, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave.
- If $k=\frac{\beta+\gamma}{\alpha+\gamma}$, which implies that $w=\bar{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is zero when $t=0$; it is positive when $0<t<T_1$; it is zero again when $t=T_1$; and it is negative when $t>T_1$. Thus, $\Psi(t|\alpha,\beta,\gamma,k)$ is still convex-concave.

So far, we have shown that when $k\le\frac{\beta+\gamma}{\alpha+\gamma}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave. Next, we consider the case that $k>\frac{\beta+\gamma}{\alpha+\gamma}$.

- When $\beta<\gamma$, we have $\underline{w}=0$. In this case, if $k>\frac{\beta+\gamma}{\alpha+\gamma}$, which implies that $\underline{w}<w<\bar{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is zero at $t=T_3$ and $t=t_4$; it is negative when $t<T_3$; it is positive when $T_3<t<T_4$; and it is negative when $t>T_4$. Therefore, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is concave-convex-concave.

When $\beta>\gamma$, we have $\underline{w}=\frac{\beta-\gamma}{\alpha-\gamma}$. In addition, $\frac{\beta+\gamma}{\alpha+\gamma}<\frac{\alpha-\gamma}{\beta-\gamma}$. There exists a k satisfying $\frac{\beta+\gamma}{\alpha+\gamma}<k<\frac{\alpha-\gamma}{\beta-\gamma}$.

- If $\frac{\beta+\gamma}{\alpha+\gamma}<k<\frac{\alpha-\gamma}{\beta-\gamma}$, which implies that $\underline{w}<w<\bar{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is zero at $t=T_3$ and $t=t_4$; it is negative when $t<T_3$; it is positive when $T_3<t<T_4$; and it is negative when $t>T_4$. Therefore, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is concave-convex-concave.
- If $k\geq\frac{\alpha-\gamma}{\beta-\gamma}$, which implies that $w\leq\underline{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is negative when $t<T_2$ and is positive when $t>T_2$. That is, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is a concave-convex function.

To wrap up, when $\alpha>\beta$, we define two thresholds for k to be

$$\underline{k}=\frac{\beta+\gamma}{\alpha+\gamma}\quad\text{and}\quad\overline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \beta>\gamma,\\ +\infty, & \beta<\gamma.\end{cases}$$

When $k\leq\underline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is convex-concave function. When $k\geq\overline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is concave-convex function. When $\underline{k}<k<\overline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is concave-convex-concave function.

We can apply a similar proof for the case that $\alpha<\beta$. The right column of Figure EC.2 displays the function $r_1(t,\alpha,\beta|\gamma)$ when $\alpha<\beta$. Similarly as above, when $\alpha<\beta$, we define two thresholds for k to be

$$\underline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha>\gamma\\ 0, & \text{if } \alpha<\gamma\end{cases}\quad\text{and}\quad\overline{k}=\frac{\beta+\gamma}{\alpha+\gamma}$$

When $k\leq\underline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is convex-concave function. When $k\geq\overline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is concave-convex function. When $\underline{k}<k<\overline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ is convex-concave-convex function. *Q.E.D.*

LEMMA EC.3. *Consider α, β, γ be three (different) positive parameters, $t\geq0$, and k be arbitrary real numbers.*

(1) Suppose $\alpha>\beta$. Recall $\overline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \beta>\gamma,\\ +\infty, & \beta<\gamma.\end{cases}$

(1.1) When $k\leq1$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ decreases and then increases in t.

(1.2) When $1<k<\overline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ increases, then decreases, and finally increases in t.

(1.3) When $k\geq\overline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ increases and then decreases in t.

(2) Suppose $\alpha<\beta$. Recall $\underline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha>\gamma.\\ 0, & \text{if } \alpha<\gamma.\end{cases}$

(2.1) When $k\leq\underline{k}$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ decreases and then increases in t.

(2.2) When $\underline{k}<k<1$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ decreases, then increases, and finally decreases in t.

(2.3) When $k \geq 1$, $\Psi(t|\alpha,\beta,\gamma,k)$ increases and then decreases in t.

Proof. We examine the first-order derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ that is given by

$$\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = k\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t} - \frac{\partial\Phi(t|\beta,\gamma)}{\partial t} = k\frac{\alpha e^{-\alpha t} - \gamma e^{-\gamma t}}{\alpha-\gamma} - \frac{\beta e^{-\beta t} - \gamma e^{-\gamma t}}{\beta-\gamma}. \tag{EC.5}$$

Suppose $k=0$. Then $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = -\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}$. Following Lemma EC.1, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases and then increases in t. In the following, we focus on the case that $k\neq 0$. First, we have $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k-1$. Hence, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = \begin{cases} \geq 0, & k\geq 1 \\ <0, & k<1 \end{cases}$.

Next, we determine the sign of $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ when t is sufficiently large. When $\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}\neq 0$, we can rewrite (EC.5) to be

$$\begin{aligned}
\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} &= k\left(\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}\right)\cdot\left\{\frac{\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t}}{\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}} - \frac{1}{k}\right\} \\
&= k\left(\frac{\beta e^{-\beta t}-\gamma e^{-\gamma t}}{\beta-\gamma}\right)\left\{\frac{(\beta-\gamma)}{(\alpha-\gamma)}\cdot\frac{(\alpha e^{-\alpha t}-\gamma e^{-\gamma t})}{(\beta e^{-\beta t}-\gamma e^{-\gamma t})} - \frac{1}{k}\right\} \\
&= k\left(\frac{\beta e^{-\beta t}-\gamma e^{-\gamma t}}{\beta-\gamma}\right)\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})} - \frac{1}{k}\right\}.
\end{aligned}$$

Lemma EC.1 indicates that $\frac{\partial\Phi(t|\beta,\gamma)}{\partial t} = \frac{\beta e^{-\beta t}-\gamma e^{-\gamma t}}{\beta-\gamma}$ is first positive and then negative. Thus, $\left(\frac{\beta e^{-\beta t}-\gamma e^{-\gamma t}}{\beta-\gamma}\right)$ must be negative when t is sufficiently large. Then we consider the ratio $\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}$. We have the following result:

$$\lim_{t\to\infty}\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})} = \begin{cases} \frac{(\beta-\gamma)}{(\alpha-\gamma)}, & \text{if } \alpha>\beta>\gamma \\ 0, & \text{if } \alpha>\gamma>\beta \\ 0, & \text{if } \gamma>\alpha>\beta \\ \frac{(\beta-\gamma)}{(\alpha-\gamma)}, & \text{if } \beta>\alpha>\gamma \\ +\infty, & \text{if } \beta>\gamma>\alpha \\ +\infty. & \text{if } \gamma>\beta>\alpha \end{cases}$$

We start with the case when $\alpha>\beta$. Recall that in this case, we defined $\underline{k} = \frac{\beta+\gamma}{\alpha+\gamma}$ and $\overline{k} = \begin{cases} \frac{\alpha-\gamma}{\beta-\gamma}, & \beta>\gamma, \\ +\infty, & \beta<\gamma. \end{cases}$. From above, we have

$$\lim_{t\to\infty}\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})} - \frac{1}{k}\right\} = \begin{cases} \frac{(\beta-\gamma)}{(\alpha-\gamma)} - \frac{1}{k}, & \beta>\gamma \\ 0-\frac{1}{k}. & \beta<\gamma \end{cases}$$

As a result, when $k<0$, $\lim_{t\to\infty}\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})} - \frac{1}{k}\right\} > 0$, implying that $\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})} - \frac{1}{k}\right\}$ is positive when t is sufficiently large. When $0<k<\overline{k}$, $\lim_{t\to\infty}\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})} - \frac{1}{k}\right\} < 0$, implying that $\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})} - \frac{1}{k}\right\}$ is negative when t is sufficiently large. When $k \geq \overline{k}$, $\lim_{t\to\infty}\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}\geq 0$, implying that $\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}$ is positive when t is sufficiently large.

Since $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}=k\left(\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}\right)\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}$ and $\left(\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}\right)$ is negative when t is sufficiently large, we conclude that when t is sufficiently large, $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is negative if $k\geq\overline{k}$ while $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is positive if $k<\overline{k}$.

So far, we have examined the sign of $\frac{\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}$ and $\lim_{t\to\infty}\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$. Below, to determine the sign of $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ for any t, we combine the results with the convexity and concavity results in Lemma EC.2. We identify four possible scenarios, as illustrated in Figure EC.3.

(a1) When $k\leq\underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}=k-1<0$ since $k\leq\underline{k}=\frac{\beta+\gamma}{\alpha+\gamma}<1$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}>0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first negative, increases in t, then becomes positive, finally decreases in t but stays positive. That is, when $k\leq\underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases and then increases in t.

(a2) When $\underline{k}<k<1$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex-concave since we have $\overline{k}>1$. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}=k-1<0$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}>0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first negative, decreases in t, then increases in t, becomes positive, and finally decreases in t but stays positive. That is, when $\underline{k}<k<1$, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases and then increases in t.

(a3) When $1\leq k<\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex-concave. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}=k-1\geq 0$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}>0$. Furthermore, when $\alpha>\beta$, we have $T_0(\alpha,\gamma)<T_0(\beta,\gamma)$. Following Lemma EC.1, $\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t}>0$ when $t<\frac{T_0(\alpha,\gamma)}{2}$ and $\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t}<0$ when $t>\frac{T_0(\alpha,\gamma)}{2}$; while $\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}>0$ when $t<\frac{T_0(\beta,\gamma)}{2}$ and $\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}<0$ when $t>\frac{T_0(\beta,\gamma)}{2}$. Therefore, for $\frac{T_0(\alpha,\gamma)}{2}<t<\frac{T_0(\beta,\gamma)}{2}$ and $k\geq 1$, we obtain $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}=k\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t}-\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}<0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis twice. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, decreases in t, becomes negative, then increases in t, becomes positive, finally decreases in t but stays positive. That is, when $1\leq k<\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first increases, then decreases in t, and finally increases in t.

(a4) When $k\geq\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}=k-1>0$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}<0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, decreases in t, then becomes negative, finally increases in t but stays negative. That is, when $k\geq\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first increases and then decreases in t.

To sum up, when $\alpha>\beta$, $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ decreases and then increases in t if $k\leq 1$; $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ increases, then decreases, and finally increases in t if $1<k<\overline{k}$; $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ increases and then decreases in t if $k\geq\overline{k}$. An illustration of the function $\Psi\left(t|\alpha,\beta,\gamma,k\right)$ can be found in Figure EC.5.

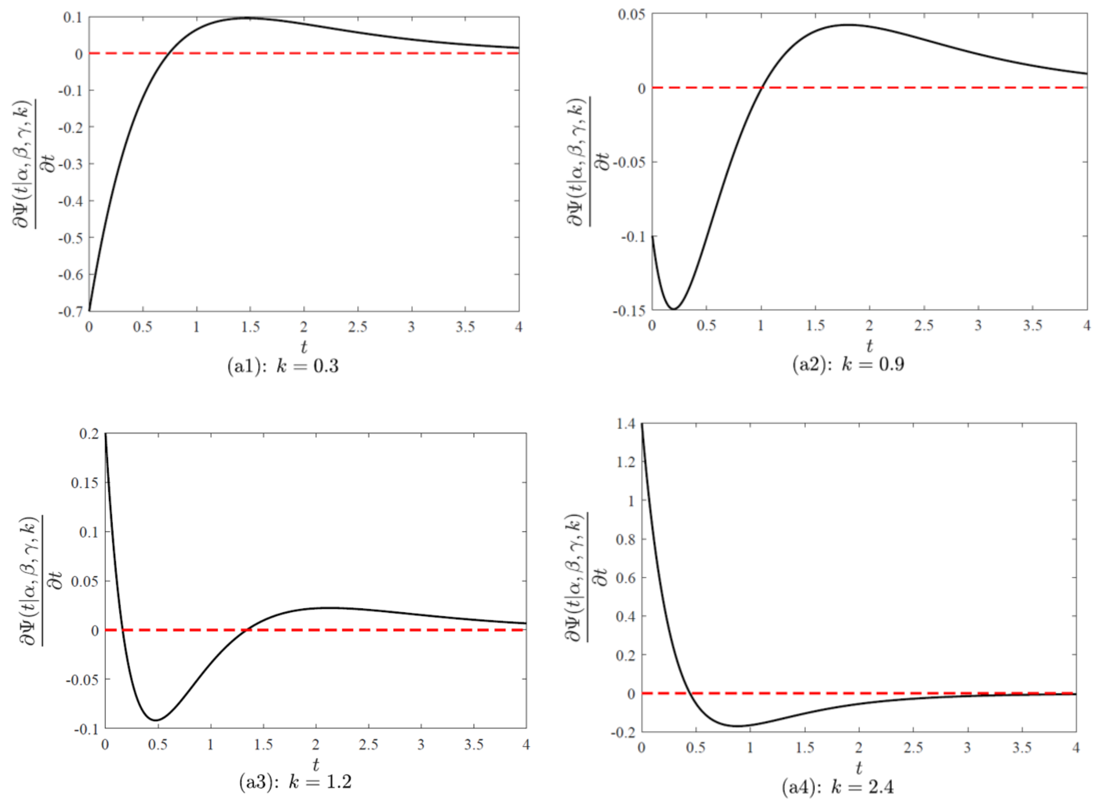


Figure EC.3 **Demonstration of the derivative** $\frac{\partial \Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ **when** $\alpha = 3 > \beta = 2$**, and** $\gamma = 1$**. Then** $\underline{k} = \frac{\beta+\gamma}{\alpha+\gamma} = 3/4$ **and** $\overline{k} = \frac{\alpha-\gamma}{\beta-\gamma} = 2$**.**

Next, we consider the case when $\alpha < \beta$. Recall that in this case, we defined $\underline{k} = \begin{cases} \frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha > \gamma \\ 0, & \text{if } \alpha < \gamma \end{cases}$ and $\overline{k} = \frac{\beta+\gamma}{\alpha+\gamma}$. Since we have

$$\lim_{t\to\infty} \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\} = \begin{cases} \frac{(\beta-\gamma)}{(\alpha-\gamma)}, & \alpha > \gamma \\ (+\infty) - \frac{1}{k}. & \alpha < \gamma \end{cases}$$

By similar analysis as above, we achieve that when $k < 0$, $\lim_{t\to\infty} \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\} > 0$. When $0 < k < \underline{k}$, $\lim_{t\to\infty} \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\} < 0$. When $k \geq \underline{k}$, $\lim_{t\to\infty} \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\} \geq 0$.

Therefore, we conclude that when t is sufficiently large, the first-order derivative $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = k\left(\frac{\partial \Phi(t|\beta,\gamma)}{\partial t}\right) \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\}$ is positive if $k \leq \underline{k}$ while $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is negative if $k > \underline{k}$.

Lastly, we combine the results about $\frac{\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}$ and $\lim_{t\to\infty} \frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ with Lemma EC.2. Again, there exist four scenarios, as illustrated in Figure EC.4.

(b1) When $k \leq \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave. In addition, $\frac{\partial \Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k - 1 < 0$ as $k \leq \underline{k} < 1$, and $\lim_{t\to\infty} \frac{\partial \Psi(t|\alpha,\beta,\gamma,k)}{\partial t} > 0$. These imply that the derivative $\frac{\partial \Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, finally stay negative, then increases in t and becomes positive, finally decreases in t but stays positive. That is, when $k \le \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases and then increases in t.

(b2) When $\underline{k} < k < 1$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave-convex since $\overline{k} >$. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k-1 < 0$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} < 0$. Furthermore, note that when $\alpha < \beta$, we have $T_0(\alpha,\gamma) > T_0(\beta,\gamma)$. Following Lemma EC.1, for $\frac{T_0(\beta,\gamma)}{2} < t < \frac{T_0(\alpha,\gamma)}{2}$ and $0 \le \underline{k} < k < 1$, we obtain $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = k\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t} - \frac{\partial\Phi(t|\beta,\gamma)}{\partial t} > 0$. Thus, the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis twice. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first negative, increases in t, becomes positive, then decreases in t, becomes negative, finally increases in t but stays negative. That is, when $\underline{k} < k < 1$, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases, then increases, and finally decreases in t.

(b3) When $1 \le k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave-convex. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k-1 \ge 0$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} < 0$. Thus, the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, increases in t, then decreases in t, becomes negative, finally increases in t but stays negative. That is, when $1 \le k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first increases, and then decreases in t.

(b4) When $k \ge \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k-1 > 0$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} < 0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, decreases in t, then becomes negative, finally increases in t but stays negative. That is, when $k \ge \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first increases, and then decreases in t.

To sum up, when $\alpha < \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ decreases and then increases in t if $k \le \underline{k}$; $\Psi(t|\alpha,\beta,\gamma,k)$ decreases, then increases, and finally decreases in t if $\underline{k} < k < 1$; $\Psi(t|\alpha,\beta,\gamma,k)$ increases and then decreases in t if $k \ge 1$. The function $\Psi(t|\alpha,\beta,\gamma,k)$ is illustrated in Figure EC.6. *Q.E.D.*

Lemma EC.2 states the convexity/concavity of $\Psi(t|\alpha,\beta,\gamma,k)$ and Lemma EC.3 indicates the monotonicity of $\Psi(t|\alpha,\beta,\gamma,k)$. Let $T_1'(\alpha,\beta,\gamma,k)$ and $T_2'(\alpha,\beta,\gamma,k)$ be the stationary points of $\Psi(t|\alpha,\beta,\gamma,k)$ when there are two points, and $T_2'(\alpha,\beta,\gamma,k)$ be the unique stationary point when there is only one stationary point. Let $T_1(\alpha,\beta,\gamma,k)$ and $T_2(\alpha,\beta,\gamma,k)$ be the inflection points of $\Psi(t|\alpha,\beta,\gamma,k)$ when there are two points, and $T_2(\alpha,\beta,\gamma,k)$ be the unique inflection point when there is only one inflection point. For simplicity, we will use T_1', T_2' , T_1, and T_2 instead. Figure EC.5 and Figure EC.6 demonstrate the function $\Psi(t|\alpha,\beta,\gamma,k)$ when $\alpha > \beta$ and when $\alpha < \beta$, respectively.

Then we prove Corollary EC.1.

Corollary EC.1. *Consider α, β, γ be three (different) positive parameters, $t \ge 0$, and k be arbitrary real numbers.*

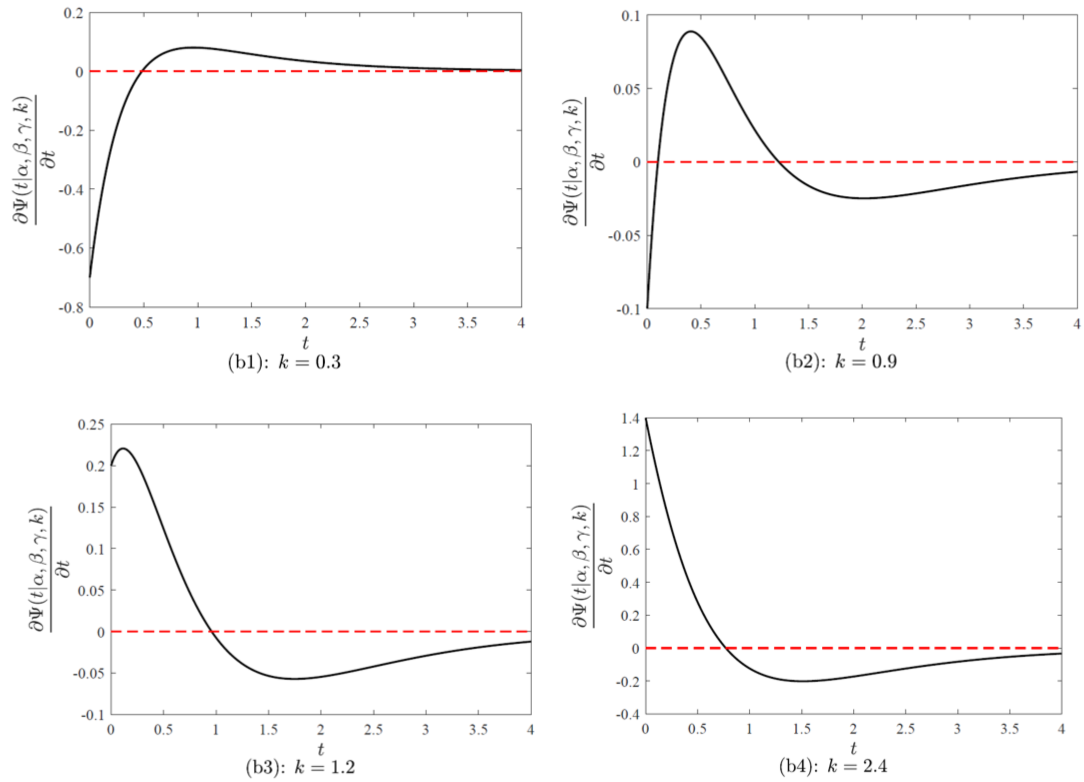


Figure EC.4 Demonstration of the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ when $\alpha = 2 < \beta = 3$, and $\gamma = 1$. Then $\underline{k} = \frac{\alpha-\gamma}{\beta-\gamma} = 1/2$ and $\overline{k} = \frac{\beta+\gamma}{\alpha+\gamma} = 4/3$.

(1) When $k \le \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave function, which decreases and then increases in t. We have $T_2' < T_2$.

(2) When $\underline{k} < k \le 1$,

(2.1) Suppose $\alpha > \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex-concave function, which decreases and then increases in t. We have $T_1 < T_2' < T_2$.

(2.2) Suppose $\alpha < \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave-convex function. $\Psi(t|\alpha,\beta,\gamma,k)$ decreases, then increases, and finally decreases in t. We have $T_1' < T_1 < T_2' < T_2$.

(3) When $1 < k < \overline{k}$,

(3.1) Suppose $\alpha > \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex-concave function. $\Psi(t|\alpha,\beta,\gamma,k)$ increases, then decreases, and finally increases in t. We have $T_1' < T_1 < T_2' < T_2$.

(3.2) Suppose $\alpha < \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave-convex function. $\Psi(t|\alpha,\beta,\gamma,k)$ increases and then decreases in t. We have $T_1 < T_2' < T_2$.

(4) When $k \ge \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex function, which increases and then decreases in t. We have $T_2' < T_2$.

Proof. The proof follows Lemma EC.2 and Lemma EC.3. *Q.E.D.*

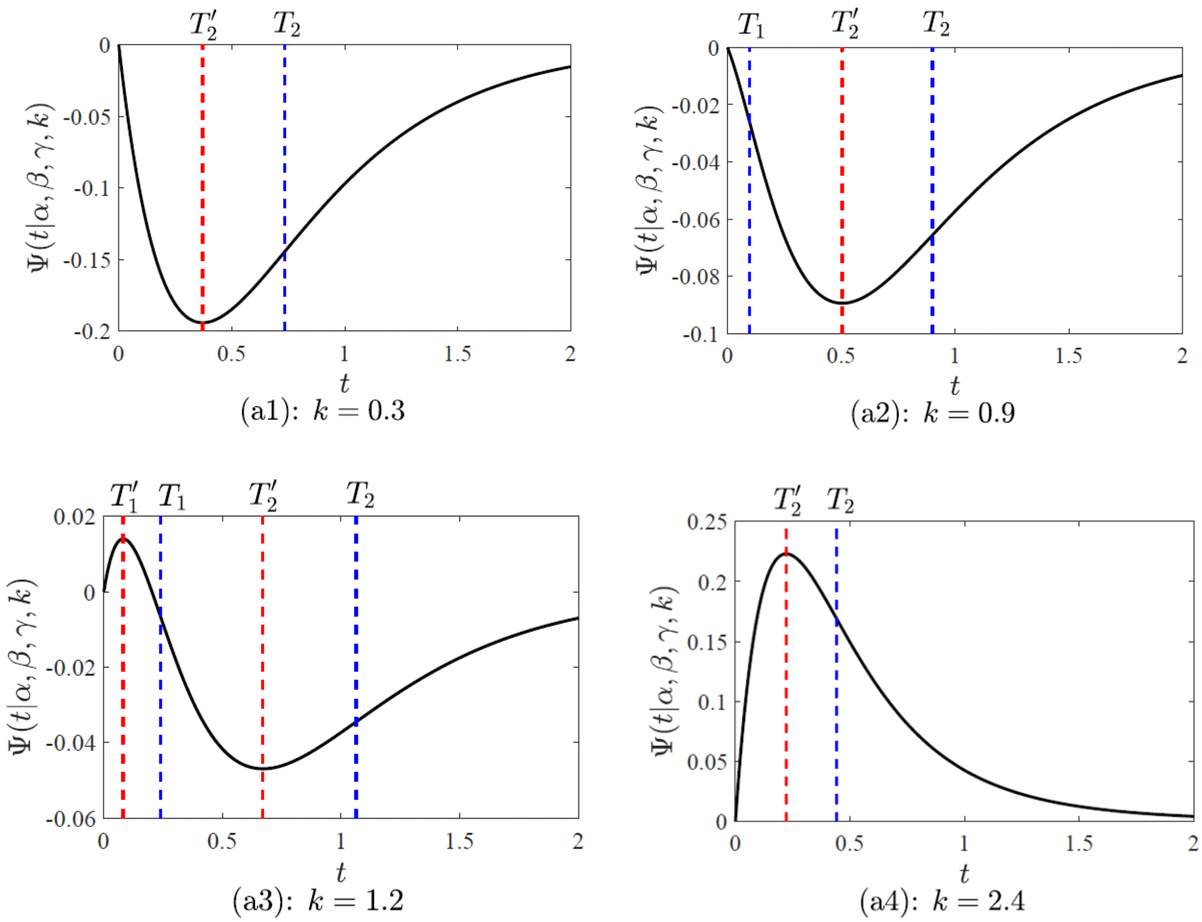


Figure EC.5 **Demonstration of the function** $\Psi(t|\alpha,\beta,\gamma,k)$ **when** $\alpha=3>\beta=2$**, and** $\gamma=1$**. Then** $\underline{k}=\frac{\beta+\gamma}{\alpha+\gamma}=3/4$ **and** $\bar{k}=\frac{\alpha-\gamma}{\beta-\gamma}=2$**.**

In the following lemma, we prove the relationship between structure and the property of function $\Psi(t|,\alpha,\beta,\gamma,k)$ in the proportional reward case.

Lemma EC.4. *In the proportional reward case,* $\Psi(t|,\alpha,\beta,\gamma,k)$ *and the optimal structure has the following relationship:*

(i) If $\Psi(t|,\alpha,\beta,\gamma,k)$ *is convex* $t\in[\bar{t}_a,\bar{t}_b]$*, where* $\bar{t}_a<\bar{t}_b$*, in the optimal schedule, elements within* $\left[(T-\bar{t}_b)^+,(T-\bar{t}_a)^+\right]$ *are in a diminuendo subsequence.*

(ii) If $\Psi(t|,\alpha,\beta,\gamma,k)$ *is concave for* $t\in[\bar{t}_a,\bar{t}_b]$*, where* $\bar{t}_a<\bar{t}_b$*, in the optimal schedule, elements within* $\left[(T-\bar{t}_b)^+,(T-\bar{t}_a)^+\right]$ *are in a crescendo subsequence.*

(iii) If $\Psi(t|,\alpha,\beta,\gamma,k)$ *is convex for* $t\in[\bar{t}_a,\bar{t}_b]$ *and concave for* $t\in[\bar{t}_b,T_c]$*, where* $\bar{t}_a<\bar{t}_b<T_c$*, in the optimal schedule if there is a game element* $\pi^*(i)$ *start before but end after* $(T-\bar{t}_b)^+$*, then we have* $r_{*(i)}\geq\min\left\{r_{*(i-1)},r_{*(i+1)}\right\}$*.*

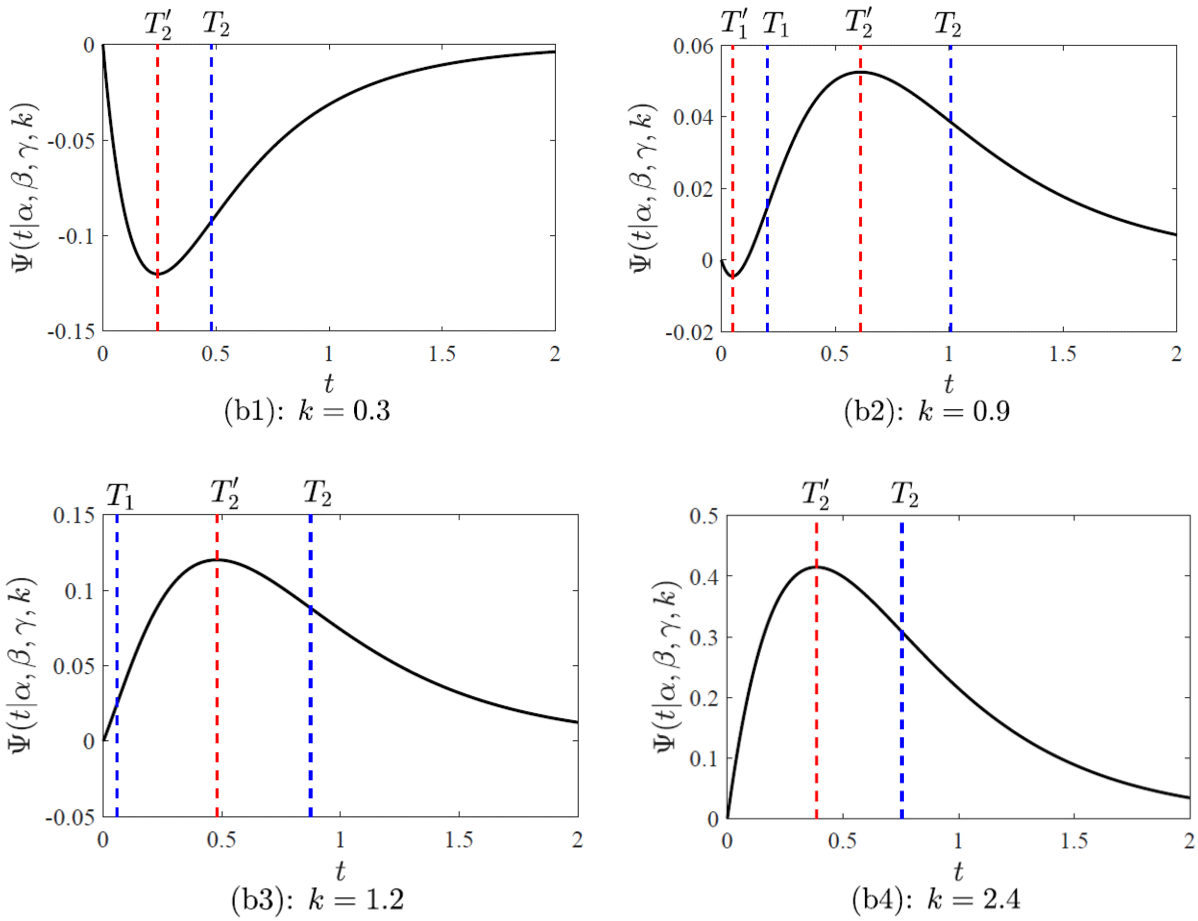


Figure EC.6 **Demonstration of the function** $\Psi(t|\alpha,\beta,\gamma,k)$ **when** $\alpha = 2 < \beta = 3$**, and** $\gamma = 1$**. Then** $\underline{k} = \frac{\alpha-\gamma}{\beta-\gamma} = 1/2$ **and** $\bar{k} = \frac{\beta+\gamma}{\alpha+\gamma} = 4/3$**.**

(iv) If $\Psi(t|,\alpha,\beta,\gamma,k)$ *is concave in* $t \in [\bar{t}_a, \bar{t}_b]$ *and convex in* $t \in [\bar{t}_b, T_c]$*, where* $\bar{t}_a < \bar{t}_b < T_c$*, in the optimal schedule if there is a game element* $\pi^*(i)$ *start before but end after* $(T-\bar{t}_b)^+$*, then we have* $r_{*(i)} \le \max\{r_{*(i-1)}, r_{*(i+1)}\}$*.*

Proof of Lemma EC.4. We use an interchange argument to prove this proposition. Let S^* be the satisfaction obtained from the optimal schedule $\boldsymbol{\pi}^*$, and let S_i^* be the satisfaction obtained by interchanging game element $\pi(i-1)^*$ and game element $\pi(i)^*$ in the optimal schedule. We have

$$S^* - S_i^* = (d_{(i-1)^*} - d_{(i)^*})\left((\Psi(\bar{t}_i + \tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1} + \tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k)) - (\Psi(\bar{t}_i|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k))\right). \tag{EC.6}$$

(i) When $\Psi(t|\alpha,\beta,\gamma,k)$ is convex for $t \in [\bar{t}_a, \bar{t}_b]$, and both game element $\pi(i-1)^*$ and game element $\pi(i)^*$ start and finish within $\left[(T-\bar{t}_b)^+, (T-\bar{t}_a)^+\right]$.

As $\Psi'(t|\alpha,\beta,\gamma,k)$ is increasing for $t \in [\bar{t}_a, \bar{t}_b]$, we have $\Psi(\bar{t}_i|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k) \le \Psi\left(\bar{t}_i + \tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1} + \tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k\right)$. By the optimality $S^* - S_i^* \ge 0$ and (EC.6), we have $d_{(i-1)^*} \ge d_{(i)^*}$.

(ii) When $\Psi(t|\alpha,\beta,\gamma,k)$ is concave for $t \in [\bar{t}_a, \bar{t}_b]$, and both game element $\pi(i-1)^*$ and game element $\pi(i)^*$ start and finish within $\left[(T-\bar{t}_b)^+, (T-\bar{t}_a)^+\right]$.

As $\Psi'(t|\alpha,\beta,\gamma,k)$ is decreasing for $t\in[\bar{t}_a,\bar{t}_b]$, we have $\Psi(\bar{t}_i|\alpha,\beta,\gamma,k)-\Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k)\geq \Psi\left(\bar{t}_i+\tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k\right)-\Psi\left(\bar{t}_{i+1}+\tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k\right)$. By the optimality, $S^*-S_i^*\geq 0$ and (EC.6), we have $d_{(i-1)^*}\leq d_{(i)^*}$.

(iii) When $\Psi(t|\alpha,\beta,\gamma,k)$ is convex for $t\in[\bar{t}_a,\bar{t}_b]$ and concave for $t\in[\bar{t}_b,T_c]$, a game element $\pi^*(i)$ starting before and finishing after $(T-\bar{t}_b)^+$ exists in the optimal schedule, where $1<i<n$. Let S^*_{i+1} be the satisfaction obtained by interchanging game element $\pi(i)^*$ and game element $\pi(i+1)^*$; then, we have

$$S^*-S^*_{i+1}=(d_{(i)^*}-d_{(i+1)^*})\left((\Psi(\bar{t}_{i+1}+\tau_{\pi(i)^*}|\alpha,\beta,\gamma,k)-\Psi(\bar{t}_{i+2}+\tau_{\pi(i)^*}|\alpha,\beta,\gamma,k))-(\Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k)-\Psi(\bar{t}_{i+2}|\alpha,\beta,\gamma,k))\right). \tag{EC.7}$$

Suppose otherwise that $d_{(i-1)^*}>d_{(i)^*}$ and $d_{(i)^*}<d_{(i+1)^*}$, by optimality $S^*-S_i^*\geq 0$ and $S^*-S^*_{i+1}\geq 0$, and by (EC.6), (EC.7) there must be $\Psi(\bar{t}_i|\alpha,\beta,\gamma,k)-\Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k)\leq \Psi\left(\bar{t}_i+\tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k\right)-\Psi\left(\bar{t}_{i+1}+\tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k\right)$ and $\Psi(\bar{t}_{i+1})-\Psi(\bar{t}_{i+2}|\alpha,\beta,\gamma,k)\geq \Psi\left(\bar{t}_{i+1}+\tau_{\pi(i)^*}|\alpha,\beta,\gamma,k\right)-\Psi\left(\bar{t}_{i+2}+\tau_{\pi(i)^*}|\alpha,\beta,\gamma,k\right)$. This contradicts the fact that $\Psi'(t|\alpha,\beta,\gamma,k)$ is increasing for $t\in[\bar{t}_a,\bar{t}_b]$ and $\Psi'(t|\alpha,\beta,\gamma,k)$ is decreasing for $t\in[\bar{t}_b,T_c]$. Therefore, $d_{*(i)}\geq \min\left\{d_{*(i-1)},d_{*(i+1)}\right\}$.

(iv) When $\Psi(t|\alpha,\beta,\gamma,k)$ is concave for $t\in[\bar{t}_a,\bar{t}_b]$ and convex for $t\in[\bar{t}_b,T_c]$, a game element $\pi^*(i)$ starting before and finishing after $(T-\bar{t}_b)^+$ exists in the optimal schedule, where $1<i<n$. Let S^*_{i+1} be the satisfaction obtained by interchanging game element $\pi(i)^*$ and game element $\pi(i+1)^*$; then we can also derive (EC.7).

Suppose otherwise that $d_{(i-1)^*}<d_{(i)^*}$ and $d_{(i)^*}>d_{(i+1)^*}$, by optimality $S^*-S_i^*\geq 0$ and $S^*-S^*_{i+1}\geq 0$, and by (EC.6), (EC.7) there must be $\Psi(\bar{t}_i|\alpha,\beta,\gamma,k)-\Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k)\geq\Psi\left(\bar{t}_i+\tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k\right)-\Psi\left(\bar{t}_{i+1}+\tau_{\pi(i-1)^*}|\alpha,\beta,\gamma,k\right)$ and $\Psi(\bar{t}_{i+1})-\Psi(\bar{t}_{i+2})\leq \Psi\left(\bar{t}_{i+1}+\tau_{\pi(i)^*}|\alpha,\beta,\gamma,k\right)-\Psi\left(\bar{t}_{i+2}+\tau_{\pi(i)^*}|\alpha,\beta,\gamma,k\right)$. This contradicts the fact that $\Psi'(t|\alpha,\beta,\gamma,k)$ is decreasing for $t\in[\bar{t}_a,\bar{t}_b]$ and $\Psi'(t|\alpha,\beta,\gamma,k)$ is increasing for $t\in[\bar{t}_b,T_c]$. Therefore, $d_{*(i)}\leq \max\left\{d_{*(i-1)},d_{*(i+1)}\right\}$. *Q.E.D.*

Proof of Theorem 1. We consider four situations to prove the proposition. By Lemma EC.2, function $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex with inflection point T_2 when $k\leq \underline{k}$; convex-concave with inflection point T_2 when $k>\underline{k}$; concave-convex-concave with inflection points T_1 and T_2 when $\alpha>\beta$ and $\underline{k}<k\leq\bar{k}$; convex-concave-convex with inflection points T_1 and T_2 when $\alpha<\beta$ and $\underline{k}<k\leq\bar{k}$.

(i) When $k\leq\bar{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex for $t\in[0,T_2]$ and concave for $t\in[T_2,+\infty]$. By Lemma EC.4, elements in $\left[0,(T-T_2)^+\right]$ are in a crescendo subsequence , and elements in $\left[(T-T_2)^+,T\right]$ are in a diminuendo subsequence . If there is a game element i starting before and ending after T_2, then we have $d_{*(i)}\geq\min\left\{d_{*(i-1)},d_{*(i+1)}\right\}$. Therefore, the optimal structure is an inverted U-shaped sequence.

(ii) When $\underline{k} < k \leq \overline{k}$ and $\alpha > \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave for $t \in [0,T_1]$, convex for $t \in [T_1,T_2]$, and concave for $t \in [T_2,+\infty]$. By Lemma EC.4, elements in $\left[0,(T-T_1)^+\right]$ are in a crescendo subsequence , elements in $\left[(T-T_1)^+,(T-T_2)^+\right]$ are in a diminuendo subsequence , and elements in $\left[(T-T_2)^+,T\right]$ are in a crescendo subsequence . If there is a game element i starting before and ending after T_1, then we have $d_{*(i)} \geq \min\left\{d_{*(i-1)}, d_{*(i+1)}\right\}$. If there is a game element j starting before and ending after T_2, then we have $d_{*(j)} \leq \max\left\{d_{*(j-1)}, d_{*(j+1)}\right\}$. Therefore, the optimal structure is a N-shaped sequence.

(iii) When $\underline{k} < k \leq \overline{k}$ and $\alpha < \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex for $t \in [0,T_1]$, concave for $t \in [T_1,T_2]$, and convex for $t \in [T_2,+\infty]$. By Lemma EC.4, elements in $\left[0,(T-T_1)^+\right]$ are in a diminuendo subsequence , elements in $\left[(T-T_1)^+,(T-T_2)^+\right]$ are in a crescendo subsequence , and elements in $\left[(T-T_2)^+,T\right]$ are in a diminuendo subsequence . If there is a game element i starting before and ending after T_1, then we have $d_{*(i)} \leq \max\left\{d_{*(i-1)}, d_{*(i+1)}\right\}$. If there is a game element j starting before and ending after T_2, then we have $d_{*(j)} \geq \min\left\{d_{*(j-1)}, d_{*(j+1)}\right\}$. Therefore, the optimal structure is an inverted N-shaped sequence.

(iv) When $k > \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave for $t \in [0,T_2]$ and convex for $t \in [T_2,+\infty]$. By Lemma EC.4, elements in $\left[0,(T-T_2)^+\right]$ are in a diminuendo subsequence, and elements in $\left[(T-T_2)^+,T\right]$ are in a crescendo subsequence . If there is a game element i starting before and ending after T_2, then we have $d_{*(i)} \leq \max\left\{d_{*(i-1)}, d_{*(i+1)}\right\}$. Therefore, the optimal structure is a U-shaped sequence. *Q.E.D.*

Proof of Theorem 2. The proof follows Theorem 1 with the situation when $T < T_2$. *Q.E.D.*

Proof of Corollary 1. By the definition of k in **??**, when $\alpha > \beta$ and $\beta > \gamma$, $\underline{k} = \frac{\beta-\gamma}{\alpha-\gamma}$ and $\overline{k} = \frac{\alpha+\gamma}{\beta+\gamma}$, we have $\underline{k} < 1 < \overline{k}$; when $\alpha > \beta$ and $\beta < \gamma$, $\underline{k} = 0$ and $\overline{k} = \frac{\alpha+\gamma}{\beta+\gamma}$, we have $\underline{k} < 1 < \overline{k}$; when $\alpha < \beta$ and $\alpha > \gamma$, $\underline{k} = \frac{\alpha+\gamma}{\beta+\gamma}$ and $\overline{k} = \frac{\beta-\gamma}{\alpha-\gamma}$, we have $\underline{k} < 1 < \overline{k}$; when $\alpha < \beta$ and $\alpha > \gamma$, $\underline{k} = \frac{\alpha+\gamma}{\beta+\gamma}$ and $\overline{k} = +\infty$, we have $\underline{k} < 1 < \overline{k}$.

To sum up, $\underline{k} < 1 < \overline{k}$ for $\alpha,\beta,\gamma > 0$ and $\alpha \neq \beta \neq \gamma$. By Theorem 1, we have the optimal sequence is N-shaped when $\alpha > \beta$ and inverted N-shaped when $\beta > \alpha$. *Q.E.D.*

Proof of Theorem 3. Before we proceed with our detailed proof, we remark that this result can be seen as a corollary of Proposition 1 in Das Gupta et al. (2016). (i) When the difficulties are fixed, the disutility introduced by the difficulty is a constant (i.e., $d\Phi(T,|\beta,\gamma)$). The only decision is the schedule of the reward, which makes it equivalent as the service provider's design problem (SPDP) in Das Gupta et al. (2016). (ii) When the rewards are fixed, the utility introduced by the reward is a constant (i.e., $r\Phi(T,|\alpha,\gamma)$). The only decision is the schedule of the difficulty, which makes the problem share same formulation as the SPDP, but a minimization rather than a maximization problem. By Proposition 1 of the Das Gupta et al. (2016), we can then prove the Theorem 3.

We use an interchange argument to prove this proposition. Let S^* be the satisfaction obtained from the optimal schedule $\boldsymbol{\pi}^*$, and let S_i^* be the satisfaction obtained by interchanging game element $\pi(i-1)^*$ and game element $\pi(i)^*$ in the optimal schedule. We prove the proposition by the situations when the difficulties are fixed and the reward are fixed, respectively.

(i) When the difficulties are fixed (i.e., $d_i = d$)

By (13), as $d\Phi(T|\beta,\gamma)$ is a constant with given parameters d, T, β, γ, we have

$$S^* - S_i^* = \left(d_{(i-1)^*} - d_{(i)^*}\right)\left(\left(\Phi\left(\bar{t}_i + \tau_{\pi(i-1)^*}|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} + \tau_{\pi(i-1)^*}|\alpha,\gamma\right)\right) - \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right)\right). \tag{EC.8}$$

We consider three cases to prove the proposition.

Case (ia): Both game element $\pi(i-1)^*$ and game element $\pi(i)^*$ start and finish within $\left[0,(T-T_0(\alpha,\gamma))^+\right]$. As $\Phi'(t|\alpha,\gamma)$ is increasing for $t\in[\min\{T_0(\alpha,\gamma),T\},T]$, we have $\Phi(\bar{t}_i|\alpha,\gamma) - \Phi(\bar{t}_{i+1}|\alpha,\gamma) \le \Phi\left(\bar{t}_i + \tau_{\pi(i-1)^*}|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} + \tau_{\pi(i-1)^*}|\alpha,\gamma\right)$. By the optimality $S^* - S_i^* \ge 0$ and (EC.6), we have $r_{(i-1)^*} \ge r_{(i)^*}$.

Case (ib): A game element $\pi^*(i)$ starting before and finishing after $(T-T_0(\alpha,\gamma))^+$ exists in the optimal schedule, where $1<i<n$. Let S_{i+1}^* be the satisfaction obtained by interchanging game element $\pi(i)^*$ and game element $\pi(i+1)^*$; then, we have

$$S^* - S_{i+1}^* = \left(r_{(i)^*} - r_{(i+1)^*}\right)\left(\left(\Phi\left(\bar{t}_{i+1} + \tau_{\pi(i)^*}|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+2} + \tau_{\pi(i)^*}|\alpha,\gamma\right)\right) - \left(\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+2}|\alpha,\gamma\right)\right)\right). \tag{EC.9}$$

Suppose otherwise that $r_{(i-1)^*} < r_{(i)^*}$ and $r_{(i)^*} > r_{(i+1)^*}$, by optimality $S^* - S_i^* \ge 0$ and $S^* - S_{i+1}^* \ge 0$, and by (EC.8), (EC.9) there must be $\Phi(\bar{t}_i|\alpha,\gamma) - \Phi(\bar{t}_{i+1}|\alpha,\gamma) \ge \Phi\left(\bar{t}_i + \tau_{\pi(i-1)^*}|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} + \tau_{\pi(i-1)^*}|\alpha,\gamma\right)$ and $\Phi(\bar{t}_{i+1}|\alpha,\gamma) - \Phi(\bar{t}_{i+2}|\alpha,\gamma) \le \Phi\left(\bar{t}_{i+1} + \tau_{\pi(i)^*}|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+2} + \tau_{\pi(i)^*}|\alpha,\gamma\right)$. This contradicts the fact that $\Phi'(t|\alpha,\gamma)$ is increasing for $t\in[\min\{T_0(\alpha,\gamma),T\},T]$ and $\Psi'(t|\alpha,\gamma)$ is decreasing for $t\in[0,\min\{T_0(\alpha,\gamma),T\}]$. Therefore, $r_{*(i)} \le \max\left\{r_{*(i-1)}, r_{*(i+1)}\right\}$.

Case (ic): Both game element $\pi(i-1)^*$ and game element $\pi(i)^*$ start and finish within $\left[(T-T_0(\alpha,\gamma))^+,T\right]$. As $\Phi'(t|\alpha,\gamma)$ is decreasing for $t\in[0,\min\{T_0(\alpha,\gamma),T\}]$, we have $\Phi(\bar{t}_i|\alpha,\gamma) - \Phi(\bar{t}_{i+1}|\alpha,\gamma) \ge \Phi\left(\bar{t}_i + \tau_{\pi(i-1)^*}|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} + \tau_{\pi(i-1)^*}|\alpha,\gamma\right)$. By the optimality $S^* - S_i^* \ge 0$ and (EC.6), we have $r_{(i-1)^*} \le r_{(i)^*}$.

(ii) When the rewards are fixed (i.e., $r_i = r$)

By (14), as $u\Phi(T|\alpha,\gamma)$ is a constant with given parameters u, T, α, γ, we have

$$S^* - S_i^* = \left(d_{(i-1)^*} - d_{(i)^*}\right)\left(\left(\Phi\left(\bar{t}_i|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\right) - \left(\Phi\left(\bar{t}_i + \tau_{\pi(i-1)^*}|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1} + \tau_{\pi(i-1)^*}|\beta,\gamma\right)\right)\right). \tag{EC.10}$$

We consider three cases to prove the proposition.

Case (iia): Both game element $\pi(i-1)^*$ and game element $\pi(i)^*$ start and finish within $\left[0,(T-T_0(\beta,\gamma))^+\right]$. As $\Phi'(t|\beta,\gamma)$ is increasing for $t\in[\min\{T_0(\beta,\gamma),T\},T]$, we have $\Phi(\bar{t}_i|\beta,\gamma)-\Phi(\bar{t}_{i+1}|\beta,\gamma)\le\Phi\left(\bar{t}_i+\tau_{\pi(i-1)^*}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+1}+\tau_{\pi(i-1)^*}|\beta,\gamma\right)$. By the optimality $S^*-S_i^*\ge 0$ and (EC.6), we have $d_{(i-1)^*}\le d_{(i)^*}$.

Case (iib): A game element $\pi^*(i)$ starting before and finishing after $(T-T_0(\beta,\gamma))^+$ exists in the optimal schedule, where $1<i<n$. Let S_{i+1}^* be the satisfaction obtained by interchanging game element $\pi(i)^*$ and game element $\pi(i+1)^*$; then, we have

$$S^*-S_{i+1}^*=\left(d_{(i)^*}-d_{(i+1)^*}\right)\left(\left(\Phi(\bar{t}_{i+1}|\beta,\gamma)-\Phi(\bar{t}_{i+2}|\beta,\gamma)\right)-\left(\Phi\left(\bar{t}_{i+1}+\tau_{\pi(i)^*}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+2}+\tau_{\pi(i)^*}|\beta,\gamma\right)\right)\right). \tag{EC.11}$$

Suppose otherwise that $d_{(i-1)^*}>d_{(i)^*}$ and $d_{(i)^*}<d_{(i+1)^*}$, by optimality $S^*-S_i^*\ge 0$ and $S^*-S_{i+1}^*\ge 0$, and by (EC.10), (EC.11) there must be $\Phi(\bar{t}_i|\beta,\gamma)-\Phi(\bar{t}_{i+1}|\beta,\gamma)\ge\Phi\left(\bar{t}_i+\tau_{\pi(i-1)^*}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+1}+\tau_{\pi(i-1)^*}|\beta,\gamma\right)$ and $\Phi(\bar{t}_{i+1}|\beta,\gamma)-\Phi(\bar{t}_{i+2}|\beta,\gamma)\le\Phi\left(\bar{t}_{i+1}+\tau_{\pi(i)^*}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+2}+\tau_{\pi(i)^*}|\beta,\gamma\right)$. This contradicts the fact that $\Phi'(t|\beta,\gamma)$ is increasing for $t\in[\min\{T_0(\beta,\gamma),T\},T]$ and $\Psi'(t|\beta,\gamma)$ is decreasing for $t\in[0,\min\{T_0(\beta,\gamma),T\}]$. Therefore, $r_{*(i)}\le\max\left\{r_{*(i-1)},r_{*(i+1)}\right\}$.

Case (ic): Both game element $\pi(i-1)^*$ and game element $\pi(i)^*$ start and finish within $\left[(T-T_0(\beta,\gamma))^+,T\right]$. As $\Phi'(t|\beta,\gamma)$ is decreasing for $t\in[0,\min\{T_0(\beta,\gamma),T\}]$, we have $\Phi(\bar{t}_i|\beta,\gamma)-\Phi(\bar{t}_{i+1}|\beta,\gamma)\ge\Phi\left(\bar{t}_i+\tau_{\pi(i-1)^*}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+1}+\tau_{\pi(i-1)^*}|\beta,\gamma\right)$. By the optimality $S^*-S_i^*\ge 0$ and (EC.6), we have $r_{(i-1)^*}\ge r_{(i)^*}$.

In summary, the following applies in the optimal schedule of this situation: (i) Elements are in a U-shaped sequence of rewards when the difficulties are fixed. (ii) Elements are in an inverted U-shaped sequence of difficulties when the rewards are fixed. *Q.E.D.*

Appendix B: Mathematical Expressions of the Optimal Structures

Table EC.1 summarizes the expressions of the optimal structure of the level design problem with proportional reward. As reward is proportional to difficulty, the difficulty sequence and reward sequence reveal the same structural properties. LDPP and LDPPR share the same structural properties. Table EC.2 summarizes the optimal structure of the level design problem with general reward.

Appendix C: Integer Programming Formulation of the LDP

The LDP defined in (12) is an integer optimization problem, but it may not be straightforward to see this because we need to enumerate all the feasible sequences. In this section, we present an alternative formulation of the LDP with a clearer integer optimization program structure.

We still use the integer decision variable $\boldsymbol{\pi}$ in (12). In addition, let $\pi^{-1}(i)$ denote the position of game element i in schedule $\boldsymbol{\pi}$, such that $\pi^{-1}(\pi(i))=i$. Furthermore, we introduce an assistant

Table EC.1 Mathematical Expressions of the Optimal Structures with Proportional Reward Scheme

Structure	Mathematical Expression
Crescendo sequence	$d_{(1)} \leq \cdots \leq d_{(n)}$.
Diminuendo sequence	$d_{(1)} \geq \cdots \geq d_{(n)}$.
U-shaped sequence	Some game element $\pi(i)$, $i \in \{1, \ldots, n\}$, exists such that $d_{(1)} \geq \cdots \geq d_{(i-1)} \geq d_{(i)} \leq d_{(i+1)} \cdots \leq d_{(n)}$.
Inverted U-shaped sequence	Some game element $\pi(i)$, $i \in \{1, \ldots, n\}$, exists such that $d_{(1)} \leq \cdots \leq d_{(i-1)} \leq d_{(i)} \geq d_{(i+1)} \cdots \geq d_{(n)}$.
N-shaped sequence	Some elements $\pi(i)$ and $\pi(j)$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $d_{(1)} \leq \cdots \leq d_{(i-1)} \leq d_{(i)} \geq d_{(i+1)} \geq \cdots \geq d_{(j-1)} \geq d_{(j)} \leq d_{(j+1)} \cdots \leq d_{(n)}$.
Inverted N-shaped sequence	Some elements $\pi(i)$ and $\pi(j)$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $d_{(1)} \geq \cdots \geq d_{(i-1)} \geq d_{(i)} \leq d_{(i+1)} \leq \cdots \leq d_{(j-1)} \leq d_{(j)} \geq d_{(j+1)} \cdots \geq d_{(n)}$.
Inverted N-shaped sequence	Some elements $\pi(i)$ and $\pi(j)$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $d_{(1)} \geq \cdots \geq d_{(i-1)} \geq d_{(i)} \leq d_{(i+1)} \leq \cdots \leq d_{(j-1)} \leq d_{(j)} \geq d_{(j+1)} \cdots \geq d_{(n)}$.

Table EC.2 Mathematical Expressions of the Optimal Structures with General Reward Scheme

Problem	Structure	Mathematical Expression
LDPGFD	U-shaped reward Sequence	Some game element $\pi(i)$, $i \in \{1, \ldots, n\}$, exists such that $r_{(1)} \geq \cdots \geq r_{[i-]} \geq r_{(i)} \leq r_{(i+1)} \leq \cdots \leq r_{[n]}$. In addition, $d_{(1)} = \cdots = d_{[n]} = d$
LDPGFR	Inverted U-shaped difficulty sequence	Some game element $\pi(i)$, $i \in \{1, \ldots, n\}$, exists such that $d_{(1)} \leq \cdots \leq d_{[i-]} \leq d_{(i)} \geq d_{(i+1)} \geq \cdots \geq d_{(n)}$. In addition, $r_{(1)} = \cdots = r_{(n)} = r$
LDPGR	HL sequence	$r_{(1)} = \cdots = r_{(n)} = r_H$ and $d_{(1)} = \cdots = d_{(n)} = d_L$
	LL-HL sequence	Some game element $\pi(i)$, $i \in \{1, \ldots, n\}$, exists such that $r_{(1)} = \cdots = r_{(i)} = r_L$ and $r_{(i+1)} = \cdots = r_{(n)} = r_H$. In addition, $d_{(1)} = \cdots = d_{(n)} = d_L$
	HH-HL sequence	$r_{(1)} = \cdots = r_{[n]} = r_H$. In addition, Some game element $\pi(i)$, $i \in \{1, \ldots, n\}$, exists such that $d_{(1)} = \cdots = d_{(i)} = d_H$ and $d_{(i+1)} = \cdots = d_{(n)} = d_L$.
	LH-LL-HL sequence	Some elements $\pi(i)$ and $\pi(j)$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $r_{(1)} = \cdots = r_{(j)} = r_L$ and $r_{(j+1)} = \cdots = r_{(n)} = r_H$; and $d_{(1)} = \cdots = d_{(i)} = d_H$ and $d_{(i+1)} = \cdots = d_{(n)} = d_L$
	LH-HH-HL sequence	Some elements $\pi(i)$ and $\pi(j)$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $r_{(1)} = \cdots = r_{(i)} = r_L$ and $r_{(i+1)} = \cdots = r_{(n)} = r_H$; and $d_{(1)} = \cdots = d_{(j)} = d_H$ and $d_{(j+1)} = \cdots = d_{(n)} = d_L$

binary variable $x_{\boldsymbol{\pi},i,j}$, which indicates the assignment between element i and the jth position of schedule $\boldsymbol{\pi}$. For a given schedule $\boldsymbol{\pi}$, we assume $x_{\boldsymbol{\pi},i,j} = 1$, if element i is the jth game element, and $x_{\boldsymbol{\pi},i,j} = 0$ otherwise. We then present the alternative formulation of the LDP as:

$$\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) \prod_{i=1}^{n} x_{\boldsymbol{\pi},i,\pi^{-1}(i)}, \tag{EC.12}$$

$$s.t., \sum_{i=1}^{n} x_{\boldsymbol{\pi},i,j} = 1, \forall j \in [n], \text{and } \boldsymbol{\pi} \in \mathcal{S}, \tag{EC.13}$$

$$\sum_{j=1}^{n} x_{\boldsymbol{\pi},i,j} = 1, \forall i \in [n], \text{and } \boldsymbol{\pi} \in \mathcal{S}, \tag{EC.14}$$

$$\boldsymbol{x} \in \{0,1\}, \tag{EC.15}$$

$$\boldsymbol{\pi} \in \mathcal{S}, \tag{EC.16}$$

where $S(\boldsymbol{\pi})$ is the remembered utility of schedule $\boldsymbol{\pi}$ defined in (12), and $\mathcal{S}$ is the set of feasible schedules. The problem (EC.12)-(EC.16) is an integer problem, and the designer has to enumerate all the feasible schedules to resolve it.

Appendix D: Parameters of the General Reward Scheme Example

Table EC.3 summarize the parameters and optimal solution of the example problem we present in Figure 5.

Table EC.3 Parameters and Optimal Schedule of the General Reward Scheme Example

Parameter	Value
Number of elements	$n = 8$
Vector of rewards	$\boldsymbol{r} = (1,2,3,4,5,6,7,8)^{\mathrm{T}}$
Vector of difficulties	$\boldsymbol{d} = (1,5,5,3,3,7,6,4)^{\mathrm{T}}$
Vector of durations	$\boldsymbol{t} = (5.90, 6.22, 3.71, 5.04, 3.16, 2.67, 5.07, 7.23)^{\mathrm{T}}$
Planning time	$T = 39.01$
Degree of reward seeking	$\alpha = 0.02$
Degree of difficulty aversion	$\beta = 0.01$
Memory-decay rate	$\gamma = 0.05$
Optimal sequence	$\boldsymbol{\pi}^* = (2,3,6,1,4,7,5,8)$

Appendix E: Extension: Repeated Use of Game Elements

In this section, we study the level design problem with repeated use of game elements. One of the distinguishing features of games is that the elements are virtual, meaning that they can be reproduced costlessly multiple times within a level. This contrasts with service design problems, like those studied in Das Gupta et al. (2016), where repeating a service element may be costly or not possible.

The possibility of repeating elements creates a new design problem that goes beyond sequencing, which has been the focus of previous papers in the literature. Here, the decision space is extended to allow the level designer to choose the number of each game element to deploy (within a given time limit) as well as how to sequence these elements. We study the optimal structure of the final sequence of game elements (allowing for repeats) in both the proportional reward and general reward settings. Proofs of the results in this section are collected in **??**

E.1 Sequencing Game Elements with Proportional-reward Scheme

In this section, we consider the level design problem, allowing for repeated elements when rewards are proportional to difficulties for all game elements. The model is the same as that studied in Section 3, except that now each of the n game elements can be used multiple times in determining a level.

We have included a few constraints on the level design problem for realism. We impose that every level design must include each of the n elements at least once. This reflects the fact that the designers of the game elements have designed them thematically to suit the level and expect them to be used at least once to contribute to the level's overall aesthetic and coherence. In addition, we impose a condition that the most difficult element can only be used once. This is the usual design aesthetic that every level should have at most one "boss", the hardest enemy or task within the level. This is important for narrative and climax.

Moreover, for tractability, we assume that the game elements have identical duration $\tau_i = \tau$ for all $i \in [n]$ and m periods, such that $m\tau = T$ and $\bar{t}_i = (m+1-i)\,\tau$. This makes it easy for different choices of game elements to add up to the duration T.Because we assume that the durations are identical, we can further simplify the LDPP since $\Psi\left(\bar{t}_i|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1}|\alpha,\beta,\gamma,k\right) = \Psi\left((m+1-i)\,\tau|\alpha,\beta,\gamma,k\right) - \Psi\left((m-i)\,\tau|\alpha,\beta,\gamma,k\right)$ is a constant with given parameters α, β, γ, k, m, i, and τ. To simplify the expression, we also define the *(potentially negative)* weight w_i of game element i as follows:

$$w_i \triangleq \left(\Psi\left((m+1-i)\,\tau|\alpha,\beta,\gamma,k\right) - \Psi\left((m-i)\,\tau|\alpha,\beta,\gamma,k\right)\right)\ \forall i \in \{1,\ldots,m\}. \tag{EC.17}$$

which encodes a notion of the value of time slot i, taking into consideration all of the associated tradeoffs. In this identical duration case, the value of the time slot i is independent of the sequence of the elements, and there can be represented by a weight for slot i. The service designer can compute the the weight w_i with given parameters before deciding the sequence of the game elements.

Then the remembered utility can be rewritten as:

$$\begin{aligned} S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} d_{(i)}\left(\Psi\left(\bar{t}_i|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1}|\alpha,\beta,\gamma,k\right)\right), \\ &= \sum_{i=1}^{m} w_i d_{(i)}. \end{aligned}$$

Using this notation, the level design problem with proportional rewards and repeated assignment (LDPPR) is:

$$\begin{aligned} \max_{\boldsymbol{\pi}}\ & S(\boldsymbol{\pi}) = \sum_{i=1}^{m} w_i d_{(i)}, \\ s.t.\ & \sum_{i=1}^{m} \mathbb{1}_{\pi(i)=j} \geq 1\ \forall j \in [n-1], \\ & \sum_{i=1}^{m} \mathbb{1}_{\pi(i)=n} = 1. \end{aligned}$$

where now $\boldsymbol{\pi}$ is an integral vector in the set $\{1,2,\ldots,n\}^m$ where $\pi(i) = j$ if element j is the ith element encountered when playing the level.

We then prove the following Lemma on the elements that are repeatedly used.

Lemma EC.5. *In the optimal solution π^* of the LDPPR, only the lowest difficulty and second highest difficulty elements (i.e., elements 1 and $n-1$) are used repeatedly. The slots assigned with element 1 have negative weights (i.e., $w_i < 0$, if $\pi(i) = 1$ for all $i \in [m]$) , and the slots assigned with element $n-1$ have positive weights (i.e., $w_i > 0$, if $\pi(i) = n-1$ for all $i \in [m]$).*

Lemma EC.5 suggests that the game designer should use a mixture of both high-difficulty and low-difficulty elements. Li et al. (2022) showed a similar result that the highest-utility and lowest-utility activities should be selected to create an ideal experiential service. The reason that a mixture of high-difficulty and low-difficulty elements are selected follows similar reasoning. Low-difficulty elements can help players relax, which resets their reference points; high-difficulty elements provide the player a challenging experience, which leaves the player with a high-intensity remembered utility contradiction.

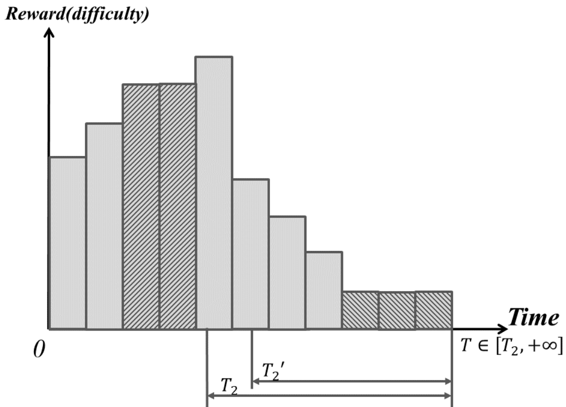
With this lemma in hand, we can prove the following proposition on the optimal structure.

Theorem EC.1. *The optimal schedule $\boldsymbol{\pi}^*$ of the LDPPR follows the same optimal structures as the LDPP, which are presented in Theorems 1 and 2.*

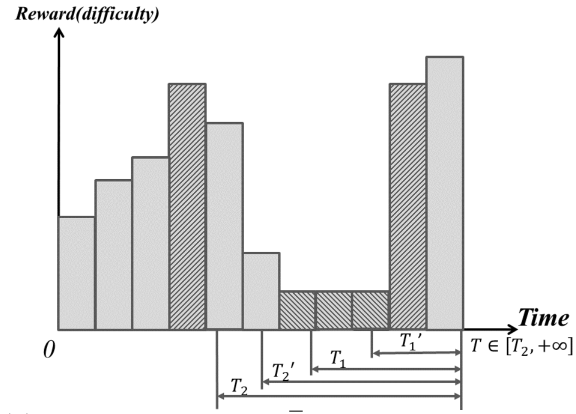
Theorem EC.1 shows that the optimal solution of LDPPR shares the same structural properties as optimal solutions to LDPP. In fact, we prove that LDPPR can be converted into an equivalent LDPP with m elements in Lemma EC.6 in Appendix A. The game designer can enhance the game experience with the repeated use of game elements, but the designer should still follow structural properties mentioned in Theorems 1 and 2 for a given type of player. Allowing for repeated use of game elements does not fundamentally alter the psychological processes of the player.

Figure EC.7 shows the possible structures of optimal schedules. The repeated elements are illustrated with a shaded bar. The tall shaded bars correspond to element $n-1$ and the short shaded bars correspond to element 1. Similar to what we saw in Figure 4, inverted U-shape, N-shape, inverted N-shape, and U-shape are optimal structures in different situations. Repeated elements are placed around the climax and low tide of the game. Element $n-1$ is used to extend the experience of a peak (e.g., Figure 7(c)), or act as a mini-boss in the middle (e.g., Figure 7(b)). Element 1 is used to extend the experience of low tide (e.g., Figure 7(b)). As discussed in the paragraph following Lemma EC.5, the repeated use of the elements $n-1$ and 1 enhances the experience because they accentuate peaks and troughs in the game experience. Element 1 resets the reference point and element $n-1$ punctuates a challenging and rewarding section of a level.

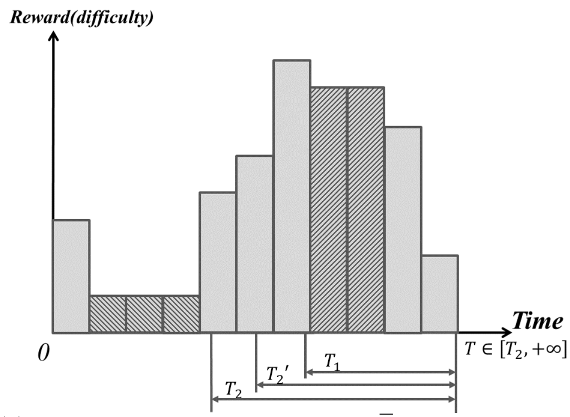
We can further prove structural results on the locations of repeated use of game elements (see Corollary EC.2 in the Appendix). This result characterizes when all of element 1 and element $n-1$ are "bunched together" (as in Figure 7(d)) or sequenced apart from each other (as is the case for the two $n-1$ elements in Figure 7(b)). The conditions are rather technical, so we leave the statement and proof of this result in Appendix A.



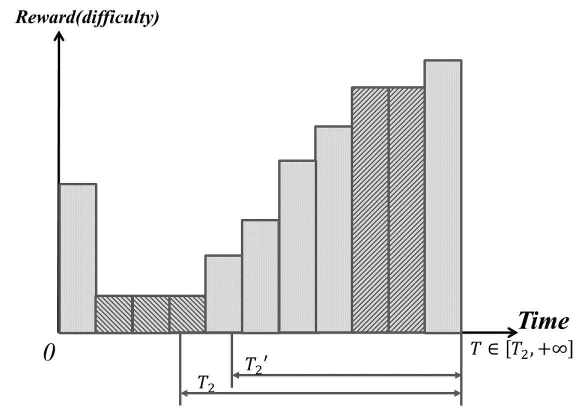
(a) Inverted U-shape when $k < \underline{k}$



(b) N-shape when $\underline{k} < k \leq \overline{k}$ and $\alpha > \beta$



(c) Inverted N-shape when $\underline{k} < k \leq \overline{k}$ and $\alpha < \beta$



(d) U-shape when $k > \overline{k}$

Figure EC.7 **An illustration of the optimal structures of the LDPPR. The repeated elements are illustrated with a shaded bar.**

E.2 Sequencing Game Elements with General-reward Scheme

In this subsection, we consider the level design problem, now allowing for the elements to be repeatedly used and with general rewards. As before, we make the game duration fixed to T for any feasible schedule, we assume that the game elements have identical duration $\tau_i = \tau$ for all $i \in [n]$ and we consider a game with m periods, such that $m\tau = T$ and $\bar{t}_i = (m+1-i)\tau$.

As this setting is quite general, we assume for tractability that there are only four types of elements. Element 1 (LL) has low reward and low difficulty (r_L and d_L), element 2 (LH) has low reward and high difficulty (r_L and d_H), element 3 (HL) has high reward and low difficulty (r_H and d_L), and element 4 (HH) has high reward and high difficulty (r_H and d_H), where $0 < r_L < r_H$ and $0 < d_L < d_H$. These four types of game elements are qualitatively representative of the possibilities one may consider in the design of a level.

The same as in Section E.1, because we assume the durations are identical, we can further simplify the LDP, as $\Phi(\bar{t}_i|\theta,\gamma) - \Phi(\bar{t}_{i+1}|\theta,\gamma) = \Phi((m+1-i)\tau|\theta,\gamma) - \Phi((m-i)\tau|\theta,\gamma)$ is a constant with given parameters α, β, γ, k, m, i, and τ. To simplify the expressions, we define the weight of the ith element as

$$w'_i(\theta) = (\Phi((m+1-i)\tau|\theta,\gamma) - \Phi((m-i)\tau|\theta,\gamma)) \ \ \forall i \in \{1,\dots,m\}. \tag{EC.18}$$

The remembered utility can be rewritten as:

$$\begin{aligned} S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} \left(r_{(i)} \left(\Phi(\bar{t}_i|\alpha,\gamma) - \Phi(\bar{t}_{i+1}|\alpha,\gamma) \right) - d_{(i)} \left(\Phi(\bar{t}_i|\beta,\gamma) - \Phi(\bar{t}_{i+1}|\beta,\gamma) \right) \right), \\ &= \sum_{i=1}^{n} \left(r_{(i)} \left(\Phi(m+1-i|\alpha,\gamma) - \Phi(m-i|\alpha,\gamma) \right) - d_{(i)} \left(\Phi(m+1-i|\beta,\gamma) - \Phi(m-i|\beta,\gamma) \right) \right), \\ &= \sum_{i=1}^{m} \left(r_{(i)} w'_i(\alpha) - d_{(i)} w'_i(\beta) \right). \end{aligned}$$

The level design problem with general reward and repeated assignment (LDPGR) can be expressed by:

$$\begin{aligned} \max_{\boldsymbol{\pi}} \ S(\boldsymbol{\pi}) &= \sum_{i=1}^{m} \left(r_{(i)} w'_i(\alpha) - d_{(i)} w'_i(\beta) \right), \\ s.t. \ \ & \pi(i) \in \{1,2,3,4\} \, \forall i \in [m]. \end{aligned}$$

Note that in this formulation we have removed the constraints (imposed in the previous subsection) that each element must be used at least once and the last element at most once. We do this for simplicity, these constraints could be added without much complication.

The following result reveals the optimal structure with general reward and repeated assignment, when the game duration is sufficiently long (i.e., $T > T'_0(\alpha,\gamma), T'_0(\beta,\gamma)$).

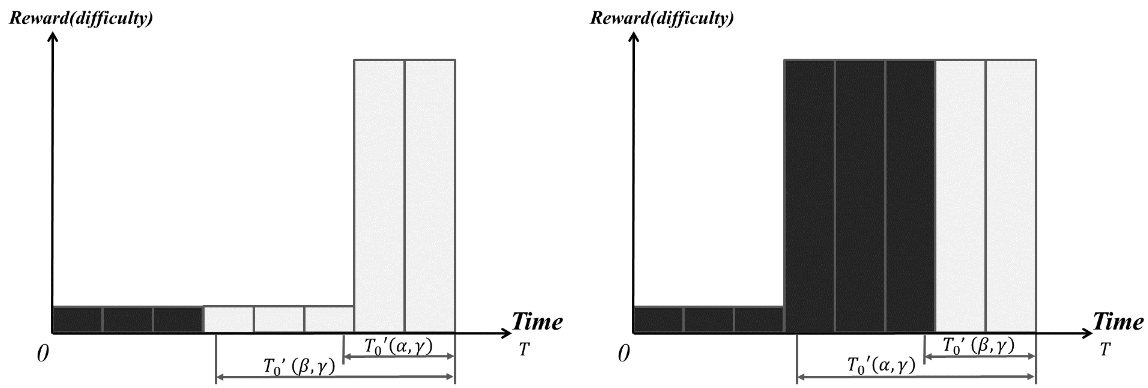
Proposition EC.1. *When the game's duration is long enough (i.e., $T > T'_0(\alpha,\gamma), T'_0(\beta,\gamma)$), in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGR, the elements' rewards (difficulties) are in the following structure.*

(i) When $T'_0(\alpha,\gamma) < T'_0(\beta,\gamma)$ (i.e., $\alpha > \beta$), the optimal structure is a LH-LL-HL sequence.

(ii) When $T'_0(\alpha,\gamma) > T'_0(\beta,\gamma)$ (i.e., $\alpha < \beta$), the optimal structure is a LH-HH-HL sequence.

Figure EC.8 helps us visualize Proposition EC.1.

The heights of the bars mark the reward of a game element. A dark-shaded bar is of high difficulty d_H a light bar is of low difficulty d_L. We can tell from Figure EC.8 that levels should begin with low-reward and high-difficulty elements and end with high-reward and low-difficulty elements. This implies that memory decay plays an important role in the player's perception. Because the



(a) LH-LL-HL shape when $\alpha > \beta$ (b) LH-HH-HL shape when $\alpha < \beta$

Figure EC.8 Optimal Structures of the LDPGR. A dark-shaded bar is of high difficulty d_H a light bar is of low difficulty d_L.

player tends to memorize the elements at the end, rewards at the end will maximize the player's remembered utility. On the other hand, difficult elements are placed at the beginning to utilize the fact that the player's mind is more clear at the moment. The results echo the analysis in Figure 5 that the service designer should value the endpoint of the game.

We further extend Proposition EC.1 by investigating the degenerate cases when the game's duration is not long enough (i.e., when $T < T_2'$) in Proposition EC.2.

Proposition EC.2. *When the game's duration is not long enough (i.e., $T < \max\{T_0'(\alpha,\gamma), T_0'(\beta,\gamma)\}$), in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGR, the elements' rewards (difficulties) are in the following structure.*

(i) The optimal structure degenerates to an LL-HL sequence when $T_0'(\alpha,\gamma) < T < T_0'(\beta,\gamma)$, and an HL sequence when $T < T_0'(\alpha,\gamma) < T_0'(\beta,\gamma)$.

(ii) The optimal structure degenerates to an HH-HL sequence when $T_0'(\beta,\gamma) < T < T_0'(\alpha,\gamma)$, and an HL sequence when $T < T_0'(\beta,\gamma) < T_0'(\alpha,\gamma)$.

The following table summarizes the optimal structures and the conditions.

Table EC.4 Optimal Structures of the LDPGR

α,β	T	Duration	Optimal Structure
	$T_0'(\alpha,\gamma) < T_0'(\beta,\gamma) < T$	Long	LH-LL-HL
$0 < \beta < \alpha$	$T_0'(\alpha,\gamma) < T < T_0'(\beta,\gamma)$	Medium	LL-HL
	$T < T_0'(\alpha,\gamma) < T_0'(\beta,\gamma)$	Short	HL
	$T_0'(\beta,\gamma) < T_0'(\alpha,\gamma) < T$	Long	LH-HH-HL
$0 < \alpha < \beta$	$T_0'(\beta,\gamma) < T < T_0'(\alpha,\gamma)$	Medium	HH-HL
	$T < T_0'(\beta,\gamma) < T_0'(\alpha,\gamma)$	Short	HL

We can tell from Table EC.4, that it is optimal to place low-reward and high-difficulty elements at the beginning, and place high-reward and low-difficulty elements at the end. The duration of the game will also affect the optimal structure.

E.3 Proof of the results in Appendix E

Proof of Lemma EC.5. It is straightforward to see that the LDPPR is an integer optimization problem with linear objective function.

(i) Suppose $\boldsymbol{\pi}^*$ is the optimal solution of the LDPPR, with repeated use of elements $j \in \{2,\dots,n-2\}$. For all the slots $i \in [m]$ in $\boldsymbol{\pi}^*$ assigned with repeated elements, we replace element $\pi(i)$ with element $n-1$ if $w_i > 0$, and with element 1 if $w_i < 0$, which formulates a schedule $\boldsymbol{\pi}_1$. There must be $S(\boldsymbol{\pi}_1) - S(\boldsymbol{\pi}^*) = \sum_{i=1}^n \left(d_{\pi_1(i)} - d_{\pi^*(i)}\right) w_i = \sum_{i=1}^n \left(d_{n-1} - d_{\pi^*(i)}\right) w_i \mathbb{1}_{w_i>0} + \sum_{i=1}^n \left(d_1 - d_{\pi^*(i)}\right) w_i \mathbb{1}_{w_i<0} > 0$, which contradicts with the assumption that $\boldsymbol{\pi}^*$ is the optimal solution. Thus, elements 1 and $n-1$, but no other elements can be repeatedly used in the optimal solution.

(ii) Suppose $\boldsymbol{\pi}^*$ is the optimal solution of the LDPPR, in which some slots with positive weight is assigned with element 1 and some slots with negative weight is assigned with element $n-1$. For all the slots in $\boldsymbol{\pi}^*$ assigned with element 1, we replace it with element $n-1$ if $w_i > 0$; For all the slots in $\boldsymbol{\pi}^*$ assigned with element $n-1$, we replace it with element $n-1$ if $w_i < 0$. This formulates a schedule $\boldsymbol{\pi}_2$. There must be $S(\boldsymbol{\pi}_2) - S(\boldsymbol{\pi}^*) = \sum_{i=1}^n \left(d_{\pi_2(i)} - d_{\pi^*(i)}\right) w_i = \sum_{i=1}^n \left(d_{n-1} - d_1\right) w_i \mathbb{1}_{w_i>0} + \sum_{i=1}^n \left(d_1 - d_{n-1}\right) w_i \mathbb{1}_{w_i<0} > 0$, which contradicts with the assumption that $\boldsymbol{\pi}^*$ is the optimal solution. Thus, if element 1 is repeated used, it is assigned with slots have negative weights; if element $n-1$ is repeated used, it is assigned with slots have positive weights. *Q.E.D.*

LEMMA EC.6. *Consider an LDPPR with a given difficulty sequence $\boldsymbol{d} = (d_1, d_2, \dots, d_n)$. Suppose $\boldsymbol{\pi}^*$ is the optimal schedule of the LDPPR, and the optimal difficulty sequence is $\boldsymbol{d}_{\pi^*} = \left(d_{\pi^*(1)}, d_{\pi^*[2]}, \dots, d_{\pi^*[m]}\right)$. Then $\boldsymbol{\pi}^*$ must also be the optimal solution of the LDPP with difficulty sequence $\boldsymbol{d}_{\pi^*}$.*

Proof of Lemma EC.6.

Suppose there is another solution $\boldsymbol{\pi}$ of the LDPP with difficulty sequence $\boldsymbol{d}_{\pi^*}$, which offers remembered utility $S(\boldsymbol{\pi}) > S(\boldsymbol{\pi}^*)$. It is straightforward to see $\boldsymbol{\pi}$ is also a feasible solution of the LDPPR with difficulty sequence $\boldsymbol{d}$. As $S(\boldsymbol{\pi}) > S(\boldsymbol{\pi}^*)$, $\boldsymbol{\pi}^*$ can not be the optimal solution of the LDPPR, which contradicts with the assumption. Therefore, $\boldsymbol{\pi}^*$ must also be the optimal solution of the LDPP with difficulty sequence $\boldsymbol{d}_{\pi^*}$. *Q.E.D.*

Proof of Theorem EC.1.

Let $\boldsymbol{\pi}^*$ be the optimal solution of the LDPPR with difficulty sequence $\boldsymbol{d}$. By Lemma EC.6, $\boldsymbol{\pi}^*$ is also the optimal solution of the LDPP with difficulty sequence $\boldsymbol{d}_{\pi^*}$. As the optimal solution of an LDPP shares the structural properties presented in Theorems 1 and 2, $\boldsymbol{\pi}^*$ must share the optimal structures presented in Theorems 1 and 2. *Q.E.D.*

To prove Corollary EC.2, we first analyze the properties of the weight $\boldsymbol{w}$ in the following lemma.

Lemma EC.7. *For the LDPPR, the weights* $\boldsymbol{w}$ *has the following properties:*

(i) If $\Psi(t|, \alpha, \beta, \gamma, k)$ *is increasing for* $t \in [\bar{t}_i, \bar{t}_{i+1}]$*, we have* $w_i > 0$.

(ii) If $\Psi(t|, \alpha, \beta, \gamma, k)$ *is decreasing for* $t \in [\bar{t}_i, \bar{t}_{i+1}]$*, we have* $w_i < 0$.

Proof of Lemma EC.7.

It is straightforward to prove this lemma by the definition of the weight in (EC.17). *Q.E.D.*

The following corollary shows some properties on the repeated assignment, which suggests repeated assignment to the climax and low tide of the game.

Corollary EC.2. *In the optimal schedule* $\boldsymbol{\pi}^*$ *of the LDPPR, the repeated used elements have the following properties.*

(1) When $k \le \underline{k}$*, if element 1 is repeated used, it is scheduled at interval* $[(T-T_2')^+, T]$*; if element* $n-1$ *is repeated used, it is scheduled at interval* $[0, (T-T_2')^+]$.

(2) When $\underline{k} \le k \le 1$,

(2.1) Suppose $\alpha > \beta$*, if element 1 is repeated used, it is scheduled at interval* $[(T-T_2')^+, T]$*, if element* $n-1$ *is repeated used, it is scheduled at interval* $[0, (T-T_2')^+]$.

(2.2) Suppose $\alpha < \beta$*, if element 1 is repeated used, it is scheduled at interval* $[0, (T-T_2')^+]$ *or* $[(T-T_1')^+, T]$*; if element* $n-1$ *is repeated used, it is scheduled at interval* $[(T-T_2')^+, (T-T_1')^+]$.

(3) When $1 < k < \overline{k}$,

(3.1) Suppose $\alpha > \beta$*, if element 1 is repeated used, it is scheduled at interval* $[(T-T_2')^+, (T-T_1')^+]$*; if element* $n-1$ *is repeated used, it is scheduled at interval* $[0, (T-T_2')^+]$ *or interval* $[(T-T_1')^+, T]$.

(3.2) Suppose $\alpha < \beta$*, if element 1 is repeated used, it is scheduled at interval* $[0, (T-T_2')^+]$*; if element* $n-1$ *is repeated used, it is scheduled at interval* $[(T-T_2')^+, T]$.

(4) When $k \ge \overline{k}$*, if element 1 is repeated used, it is scheduled at interval* $[0, (T-T_2')^+]$*; if element* $n-1$ *is repeated used, it is scheduled at interval* $[(T-T_2')^+, T]$.

Proof of Corollary EC.2. By Corollary EC.1 and Lemma EC.7, we have the following properties on the value of the weight.

(1) When $k \le \underline{k}$, elements in $[(T-T_2')^+, T]$ have negative weights, and elements in $[0, (T-T_2')^+]$ have positive weights.

(2) When $k \le 1$,

(2.1) Suppose $\alpha > \beta$, elements in $[(T-T_2')^+, T]$ have negative weights; elements in $[0, (T-T_2')^+]$ have positive weights.

(2.2) Suppose $\alpha < \beta$, elements in $[0, (T - T_2')^+]$ or $[(T - T_1')^+, T]$ have negative weights; elements in $[(T - T_2')^+, (T - T_1')^+]$ have positive weights.

(3) When $1 < k < \overline{k}$,

(3.1) Suppose $\alpha > \beta$, elements in $[(T - T_2')^+, (T - T_1')^+]$ have negative weights; elements in $[0, (T - T_2')^+]$ or $[(T - T_1')^+, T]$ have positive weights.

(3.2) Suppose $\alpha < \beta$, elements in $[0, (T - T_2')^+]$ have negative weights; elements in $[(T - T_2')^+, T]$ have positive weights.

(4) When $k \geq \overline{k}$, elements in $[0, (T - T_2')^+]$ have negative weights; elements in $[(T - T_2')^+, T]$ have positive weights.

By Lemma EC.7, element 1 can be assigned to slots with negative weights, and element $n-1$ can be assigned to slots with positive weights. We thus prove the Corollary EC.2. *Q.E.D.*

We then prove the optimal structural property of the LDPGR.

Lemma EC.8. *In the optimal solution π^* of the LDPGR, for the ith element, if both $w_i'(\alpha)$ and $w_i'(\beta)$ are positive, then element HL is assigned to slot i; if both $w_i'(\alpha)$ is positive and $w_i'(\beta)$ is negative, then element HH is assigned to slot i; if both $w_i'(\alpha)$ is negative and $w_i'(\beta)$ is positive, then element LL is assigned to slot i; if both $w_i'(\alpha)$ and $w_i'(\beta)$ are negative, then element LH is assigned to slot i.*

Proof of Lemma EC.8. Let $\boldsymbol{\pi}_1$ be the schedule in which element HL is assigned to slots with positive weights $\boldsymbol{w}'(\alpha)$ and $\boldsymbol{w}'(\beta)$; element HH is assigned to slots with positive weight $\boldsymbol{w}'(\alpha)$ and negative weight $\boldsymbol{w}'(\beta)$; element LL is assigned to slots with negative weight $\boldsymbol{w}'(\alpha)$ and positive weight $\boldsymbol{w}'(\beta)$; element LH is assigned to slots with negative weights $\boldsymbol{w}'(\alpha)$ and $\boldsymbol{w}'(\beta)$. For any element-slot assignment allowing repeated usage $\boldsymbol{\pi}_2 \neq \boldsymbol{\pi}_1$, there must be $S(\boldsymbol{\pi}_1) - S(\boldsymbol{\pi}_2) = \sum_{i=1}^{n} \left(\left(r_{\pi_1(i)} - r_{\pi_2(i)} \right) w_i'(\alpha) - \left(d_{\pi_1(i)} - d_{\pi_2(i)} \right) w_i'(\beta) \right) > 0$. Because there must be $r_{\pi_1(i)} - r_{\pi_2(i)} = r_H - r_{\pi_2(i)} \geq 0$ when $w_i'(\alpha) > 0$; $r_{\pi_1(i)} - r_{\pi_2(i)} = r_L - r_{\pi_2(i)} \leq 0$ when $w_i'(\alpha) < 0$; $d_{\pi_1(i)} - d_{\pi_2(i)} = d_L - r_{\pi_2(i)} \leq 0$ when $w_i'(\beta) > 0$; $d_{\pi_1(i)} - d_{\pi_2(i)} = d_H - r_{\pi_2(i)} \geq 0$ when $w_i'(\beta) < 0$. *Q.E.D.*

Next, we study the property with the relationship between function $\Phi(\cdot)$ and the optimal subsequence.

Corollary EC.3. *In the optimal solution π^* of the LDPGR, if both $\Phi(\alpha)$ and $\Phi(\beta)$ are decreasing in for $t \in [\bar{t}_{i+1}, \bar{t}_i]$, then the element LH is assigned to slot i; if $\Phi(\alpha)$ is decreasing and $\Phi(\beta)$ is increasing in for $t \in [\bar{t}_{i+1}, \bar{t}_i]$, then the element LL is assigned to slot i; if $\Phi(\alpha)$ is increasing and $\Phi(\beta)$ is decreasing in for $t \in [\bar{t}_{i+1}, \bar{t}_i]$, then the element HH is assigned to slot i; if both $\Phi(\alpha)$ and $\Phi(\beta)$ are increasing in for $t \in [\bar{t}_{i+1}, \bar{t}_i]$, then the element HL is assigned to slot i.*

Proof of Corollary EC.3. By Lemma EC.8 and the definition of the weight in (EC.18), we have the following properties:

(i) In $t \in [\bar{t}_{i+1}, \bar{t}_i]$, if both $\Phi(t|\alpha,\gamma)$ and $\Phi(t|\beta,\gamma)$ are decreasing, then $w_i'(\alpha) < 0$ and $w_i'(\alpha) < 0$. Thus, element LH should be assigned to slot i.

(ii) In $t \in [\bar{t}_{i+1}, \bar{t}_i]$, if $\Phi(t|\alpha,\gamma)$ is decreasing and $\Phi(t|\beta,\gamma)$ is increasing, then $w_i'(\alpha) < 0$ and $w_i'(\beta) > 0$. Thus, element LL should be assigned to slot i.

(iii) In $t \in [\bar{t}_{i+1}, \bar{t}_i]$, if $\Phi(t|\alpha,\gamma)$ is increasing and $\Phi(t|\beta,\gamma)$ is decreasing, then $w_i'(\alpha) > 0$ and $w_i'(\beta) < 0$. Thus, element HH should be assigned to slot i.

(iv) In $t \in [\bar{t}_{i+1}, \bar{t}_i]$, if both $\Phi(t|\alpha,\gamma)$ and $\Phi(t|\beta,\gamma)$ are increasing, then $w_i'(\alpha) > 0$ and $w_i'(\alpha) > 0$. Thus, element HL should be assigned to slot i. *Q.E.D.*

Then we prove the optimal structure of the level design problem with general rewards and repeated assignment.

Proof of Proposition EC.1. (i) When $\alpha > \beta$ (or when $T_0'(\alpha,\gamma) < T_0'(\beta,\gamma)$ by (EC.1)), by Lemma EC.1, there exist slots i and j $(i < j)$ where $\bar{t}_{i+2} < T_0'(\beta) < \bar{t}_{i+1}$ and $\bar{t}_{j+2} < T_0'(\alpha) < \bar{t}_{j+1}$, such that weight $\boldsymbol{w}'(\alpha)$ and weight $\boldsymbol{w}'(\beta)$ of slots $\{1,\ldots,i\}$ are negative, $\boldsymbol{w}'(\alpha)$, and weight $\boldsymbol{w}'(\beta)$ of slot $i+1$ can be either positive or negative; weight $\boldsymbol{w}'(\alpha)$ of slots $\{i+2,\ldots,j\}$ are negative, weight $\boldsymbol{w}'(\beta)$ of slots $\{i+2,\ldots,j\}$ are positive, $\boldsymbol{w}'(\alpha)$, and weight $\boldsymbol{w}'(\beta)$ of slot $j+1$ can be either positive or negative; weight $\boldsymbol{w}'(\alpha)$ and $\boldsymbol{w}'(\beta)$ of slots $\{i+2,\ldots,j\}$ are positive. By Corollary EC.3, in the optimal schedule, slots $\{1,\ldots,i\}$ are assigned with element LH; slots $\{i+2,\ldots,j\}$ are assigned with element LL; slots $\{j+2,\ldots,n\}$ are assigned with element HL; slot $i+1$ is assigned with either LH or LL;slot $j+1$ is assigned with either LL or HL.

(ii) When $\alpha < \beta$ (or when $T_0'(\alpha,\gamma) > T_0'(\beta,\gamma)$ by (EC.1)), by Lemma EC.1, there exist slots i and j $(i < j)$ where $\bar{t}_{i+2} < T_0'(\alpha) < \bar{t}_{i+1}$ and $\bar{t}_{j+2} < T_0'(\beta) < \bar{t}_{j+1}$, such that weight $\boldsymbol{w}'(\alpha)$ and weight $\boldsymbol{w}'(\beta)$ of slots $\{1,\ldots,i\}$ are negative, $\boldsymbol{w}'(\alpha)$, and weight $\boldsymbol{w}'(\beta)$ of slot $i+1$ can be either positive or negative; weight $\boldsymbol{w}'(\alpha)$ of slots $\{i+2,\ldots,j\}$ are positive, weight $\boldsymbol{w}'(\beta)$ of slots $\{i+2,\ldots,j\}$ are negative, $\boldsymbol{w}'(\alpha)$, and weight $\boldsymbol{w}'(\beta)$ of slot $j+1$ can be either positive or negative; weight $\boldsymbol{w}'(\alpha)$ and $\boldsymbol{w}'(\beta)$ of slots $\{i+2,\ldots,j\}$ are positive. By Corollary EC.3, in the optimal schedule, slots $\{1,\ldots,i\}$ are assigned with element LH; slots $\{i+2,\ldots,j\}$ are assigned with element HH; slots $\{j+2,\ldots,n\}$ are assigned with element HL; slot $i+1$ is assigned with either LH or HH;slot $j+1$ is assigned with either HH or HL.

Therefore, (i) When $\alpha > \beta$, the optimal structure is a LH-LL-HL sequence. (ii) When $\alpha < \beta$, the optimal structure is a LH-HH-HL sequence. *Q.E.D.*

Proof of Proposition EC.2. The proof follows Proposition EC.1 with the situation when $T < T_0'$. *Q.E.D.*

We then prove the following lemma regarding coefficient δ, δ_r and δ_d and the optimal structure

LEMMA EC.9. *When $\delta \neq 1$, $\delta_r \neq 1$ and $\delta_d \neq 1$ we have:*

(i) Theorems 1 to 2 and Lemma EC.5 still hold with reward ratio $k' = \frac{\delta_r}{\delta_d} k$ and difficulty-aversion degree $\beta' = \delta\beta$, where k is the reward ratio.

(ii) Theorem 3, Propositions EC.1 and EC.2 still hold for any positive constants δ_r and δ_d, and the degree of difficulty-aversion $\beta' = \delta\beta$.

Proof of Lemma EC.9. When $\delta \neq 1$, $\delta_r \neq 1$ and $\delta_d \neq 1$, we have

$$
\begin{aligned}
S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} v(t)\, e^{-\gamma(T-t)} dt, S(\boldsymbol{\pi}), \\
&= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\delta_r u_r(t) - \delta_d u_d(t)\right) e^{-\gamma(T-t)} dt, \\
&= \delta_r \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\left(r_{(1)} - f(0)\right) e^{-\alpha \bar{t}} + \sum_{j=2}^{i} \left(r_{(j)} - r_{(j-1)}\right) e^{-\alpha\left(\bar{t} - \bar{t}_{j-1}\right)} \right) e^{-\gamma(T-t)} dt \\
&\quad - \delta_d \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\left(d_{(1)} - g(0)\right) e^{-\beta' \bar{t}} + \sum_{j=2}^{i} \left(d_{(j)} - d_{(j-1)}\right) e^{-\beta'\left(\bar{t} - \bar{t}_{j-1}\right)} \right) e^{-\gamma(T-t)} dt, \\
&= \delta_r \sum_{i=1}^{n} r_{(i)} \left(\Phi\left(\bar{t}_i | \alpha, \gamma\right) - \Phi\left(\bar{t}_{i+1} | \alpha, \gamma\right)\right) - \delta_d \sum_{i=1}^{n} d_{(i)} \left(\Phi\left(\bar{t}_i | \beta', \gamma\right) - \Phi\left(\bar{t}_{i+1} | \beta', \gamma\right)\right)
\end{aligned}
$$

(i) When we consider a proportional reward, let $k' = \frac{\delta_r}{\delta_d} k$ the player's remembered utility with proportional reward can be expressed by:

$$
\begin{aligned}
S(\boldsymbol{\pi}) &= \delta_d \left(\sum_{i=1}^{n} k' d_{(i)} \left(\Phi\left(\bar{t}_i | \alpha, \gamma\right) - \Phi\left(\bar{t}_{i+1} | \alpha, \gamma\right)\right) - \sum_{i=1}^{n} d_{(i)} \left(\Phi\left(\bar{t}_i | \beta', \gamma\right) - \Phi\left(\bar{t}_{i+1} | \beta', \gamma\right)\right) \right), \\
&= \delta_d \sum_{i=1}^{n} d_{(i)} \left(\Psi\left(\bar{t}_i | \alpha, \beta', \gamma, k'\right) - \Psi\left(\bar{t}_{i+1} | \alpha, \beta', \gamma, k'\right)\right).
\end{aligned}
$$

By the proofs of Theorems 1 to 2 the optimal structure are the same as mentioned in Theorems 1 to 2 with reward ratio k' and difficulty-aversion degree β'. By the proof of Lemma EC.5 the selection of the repeated used elements are the same as mentioned in Lemma EC.5 depending on the reward ratio k' and difficulty-aversion degree β'.

(ii) When we consider a general reward scheme, and the elements share a fixed reward $r_i = r$ for all $i \in [n]$ the LDPGFD can be expressed by:

$$
\begin{aligned}
\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) &= \delta_r \sum_{i=1}^{n} r_{(i)} \left(\Phi\left(\bar{t}_i | \alpha, \gamma\right) - \Phi\left(\bar{t}_{i+1} | \alpha, \gamma\right)\right) - \delta_d \sum_{i=1}^{n} d \left(\Phi\left(\bar{t}_i | \beta', \gamma\right) - \Phi\left(\bar{t}_{i+1} | \beta', \gamma\right)\right), \\
&= \delta_r \sum_{i=1}^{n} r_{(i)} \left(\Phi\left(\bar{t}_i | \alpha, \gamma\right) - \Phi\left(\bar{t}_{i+1} | \alpha, \gamma\right)\right) - \delta_d d \Phi\left(T | \beta', \gamma\right).
\end{aligned}
$$

When the elements share a fixed reward $r_i = r$ for all $i \in [n]$ the level design problem with general reward and fixed reward (LDPGFR) can be expressed by

$$
\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) = \delta_r \sum_{i=1}^{n} r \left(\Phi\left(\bar{t}_i | \alpha, \gamma\right) - \Phi\left(\bar{t}_{i+1} | \alpha, \gamma\right)\right) - \delta_d \sum_{i=1}^{n} d_{(i)} \left(\Phi\left(\bar{t}_i | \beta', \gamma\right) - \Phi\left(\bar{t}_{i+1} | \beta', \gamma\right)\right),
$$

$$= \delta_r r \Phi\left(T|\alpha,\gamma\right) + \sum_{i=1}^{n} d_{(i)}\left(\Phi\left(\bar{t}_{i+1}|\beta',\gamma\right) - \delta_d \Phi\left(\bar{t}_i|\beta',\gamma\right)\right).$$

Given that δ_d and δ_r are positive constants, by the proof of Theorem 3, the optimal structure is exactly the same as mentioned in Theorem 3 with difficulty-aversion degree β'. By the same reason, the properties presented in Propositions EC.1 and EC.2 still hold with difficulty-aversion degree β' by their proofs. *Q.E.D.*

Appendix F: Calibration of the model

In this section, we discuss how to calibrate the parameters in the model with actual game design data from Mauro (2019) on the game design-inspired *Mario Maker 2* title. In *Mario Maker 2*, players can design "mini-levels" to be attempted by other players. Statistics on how hard and satisfying the levels were captured by Nintendo (the game's publisher) and hosted on the website `supermariomakerbookmark.nintendo.net`. We are interested in the level design problem of combining a sequence of these "mini-levels" (our game elements) into a complete level of gameplay.

Data calibration problem

We assume that the game designer is focused on a specific target player group, such that the players have homogeneous parameters α, β, and γ in the gameplay. Suppose the rewards $\boldsymbol{r}$ and difficulties $\boldsymbol{d}$ are given, and the game designer has extracted the players' satisfaction S' over the game. Then, data calibration problem (DCP) is to minimize the square error between the real satisfaction S' and the remembered utility of the level design problem (12):

$$\min_{\alpha,\beta,\gamma} \sum_{i=1}^{m} \left(S(\alpha,\beta,\gamma|\boldsymbol{\pi}) - S_i'\right)^2, \tag{DCP-1}$$

$$s.t.\ \alpha,\beta,\gamma \geq 0. \tag{DCP-2}$$

Observe that (DCP) is a nonlinear minimization problem on the error between the remembered utility $S(\alpha,\beta,\gamma|\pi_i)$ and the sample remembered utility S'.

Game design and data acquisition.

To calibrate the parameters for the targeted customers, the game designer needs to first design the game elements. As pointed out by Mauro (2019) for a sample i, the reward $\boldsymbol{r}_i$ can be obtained by asking the targeted players to rate the element after playing the game, and the remembered utility of the sample S_i' can be acquired by asking the players to rate the game after completing the whole game sequence. We normalize the rating into the discrete number from 0.1 to 1.0. The difficulty of the game elements $\boldsymbol{d}_i$ can be estimated by computing the clear rate, which is the average times one player need to take for a single pass of an element over the maximum number of attempts.

An example data set with nine game elements (i.e., "mini-levels") and five sequences is shown in Table EC.5. The reward and difficulty are acquired from Mauro (2019). Because the dataset does not contain the information on the schedule of the game elements and the remembered utility of a sequence, we just provide an example sample here to illustrate the method for the parameter calibration.

Table EC.5 An example dataset for the calibration

Sample i	$\boldsymbol{r}_i$	$\boldsymbol{d}_i$
1-5	$(0.2, 0.2, 0.1, 0.1, 0.1, 0.2, 0.1, 0.1, 0.9)^{\mathrm{T}}$	$(0.017, 0.021, 0.035, 0.050, 0.063, 0.155, 0.193, 0.400, 0.860)^{\mathrm{T}}$
Sample i	$\boldsymbol{\pi}_i$	$\boldsymbol{S'}_i$
1	$(1, 2, 3, 4, 5, 6, 7, 8, 9,)^{\mathrm{T}}$	0.7
2	$(9, 8, 7, 6, 5, 4, 3, 2, 1,)^{\mathrm{T}}$	0.3
3	$(1, 3, 5, 7, 9, 8, 6, 4, 2)^{\mathrm{T}}$	0.6
4	$(8, 6, 4, 2, 1, 3, 5, 7, 9,)^{\mathrm{T}}$	0.5
5	$(1, 3, 5, 7, 6, 4, 2, 8, 9)^{\mathrm{T}}$	0.9

Optimizing for calibration

We solve the DCP with the MATLAB `fmincon` solver. Plugging in the data in Table EC.5, we obtain the calibrated parameters: $\alpha = 0.036$, $\beta = 0.537$, and $\gamma = 0.001$ with multiple trail of different initial solutions. These parameter setting can then be used to find an optimal level design.

It should be noted that this calibration is a proof-of-concept for an actual calculation a game designer would use. In particular, we simulated the satisfaction data $S\prime_i$, where in practice this would need to be obtained by playtesters.

Appendix G: A description of the model for more general interactive services

A designer of an interactive service must sequence n elements into a complete experience. For example, consider a head summer camp counselor who must choose from among different activities (swimming, crafts, survival training, hiking, etc.) to design a program for the campers.

Each element has a reward r_i, duration τ_i, and a difficulty level d_i. These values are given to the designer. For example, the head summer camp counselor much choose among pre-designed activities (walk up a hill and back, learn to make a fire) each of which has a set difficulty (learning to make a fire is harder than going on a short hike) provides rewards to campers upon their completion (could be in terms of external rewards like badges but also intrinsic rewards like a sense of excitement or accomplishment). The same processes of accomplishment and stress can be defined as in Section 3 lead to the design problem (LDPP), assuming rewards and difficulties are proportional. Under different assumptions, we get the various design problems discussed in the paper.

The results also apply to this setting. For example, an N-shape design starts the campers out with an easy activity (e.g., hiking to a campsite), leading up to a challenging but rewarding activity (e.g., starting a fire without using a match in order to make lunch), followed by a sequence of easier tasks (e.g., eating lunch), leading to a crescendo of more challenging activities (e.g., swimming to an island lake and back). The optimal sequence of activities depends on the model parameters α, β, and γ that characterize the nature of the campers.